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Solution of Flows in Perfect Fluid Space-Time

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Abstract

Our investigation revolves around examining solutions within the context of Ricci and Einstein flows in perfect fluid space-time. We specifically focus on scenarios where the quasi-conformal and pseudo-projective curvature tensors in perfect fluid space-time exhibit pseudo-symmetry and flatness. Additionally, we provide conditions that determine whether the solutions of the Ricci and Einstein flows manifest as steady, expanding, or shrinking.

Keywords: Einstein Soliton, Lorentz Space, Perfect Fluid, Ricci and Einstein Flows

1.0 Introduction

In modern physics, space and time are inseparable, at least in the process of representing physical things through ourselves, where these two dimensions play an important role in imagining and conceptualizing the connections of all physical things. Einstein introduced general relativity in 1915, which exposes the fundamental role between the physics and geometry of spacetime. It plays an important role in Engineering and mining when applied to day to day life.

In the vast expanse of space, where distances are measured in light-years and time bends under the influence of gravity, the concept of mining takes on a whole new dimension. Here, amidst celestial bodies and cosmic phenomena, the utilization of mines transcends earthly confines, extending into the realm of space-time manifold. Mines, traditionally associated with terrestrial extraction of resources, find themselves repurposed and reimagined within the intricate fabric of space-time. In this multifaceted landscape, they serve as instruments of exploration, resource acquisition, and even defense, reshaping our understanding of both mining practices and the utilization of space. Within the depths of asteroids, moons, and planetary bodies, mines are deployed to extract valuable minerals, rare elements, and precious metals. These extraterrestrial resources hold the promise of fueling future space endeavors, sustaining human colonies, and powering advanced technologies that propel us deeper into the cosmos. Moreover, mines in the context of space-time manifold play a pivotal role in scientific endeavors, enabling the study of cosmic phenomena and the unraveling of the universe's mysteries. By strategically placing sensor-equipped mines, scientists can monitor gravitational waves, map the distribution of dark matter, and observe the dynamics of black holes, shedding light on the fundamental workings of the cosmos.Yet, amidst the marvels of space mining, challenges abound. The hostile environment of space presents formidable obstacles, from extreme temperatures to cosmic radiation, testing the resilience of both equipment and personnel. Moreover, ethical considerations loom large, as the exploitation of extraterrestrial resources raises questions about sustainability, equity, and the preservation of celestial ecosystems. As we venture further into the uncharted territories of space-time manifold, the utilization of mines stands as a testament to human ingenuity and our enduring quest for exploration and discovery. It is a frontier where

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science, technology, and imagination converge, shaping the future of humanity's cosmic journey.

The 4−dimensional vector space of the flat Minkowski spacetime can be represented as *M*. In the framework of general relativity, we examine space-time as a 4-dimensional pseudo-Riemannian manifold denoted by (*M*, *g*), where g represents a perfectly liquid space-time with a Lorentzian metric. Einstein's equation express the functioning of a perfect fluid inside a spherical entity. A perfect fluid represents as isotropic and star-like in its rest frame. Dust serves as a straightforward example of a perfect fluid. In the realm of general relativity, perfect fluids are frequently employed to represent idealized distributions. Fuels are composed of molecules that maintain contact but can easily slide over one another without encountering shear stresses. A perfect fluid, exemplified by oil, exhibits properties closely resembling those of fuels.

Viscous fluids consist of molecules that are like those of the perfect fluid. Fluids that require consideration of viscosity, or internal friction, are termed viscous fluids. Newtonian fluids are a specific category within viscous fluids, characterized by constant viscosity across varying rates of shear and with no temporal changes.

Further, Hamilton[1] introduced Ricci flow, while Catino and Mazzieri[2] introduced Einstein flow. These constitute partial differential equations governing Riemannian manifolds in any dimension, with solitons as their fixed points. Einstein and Ricci solitons, producing solutions equivalent to Einstein and Ricci flows, are delineated by:

$$\frac{\partial g}{\partial t} = -2\left(S - \frac{scal}{2}g\right) \quad and \quad \frac{\partial g}{\partial t} = -2S. \qquad (1)$$

In 2009, Cho and Kimura[3] first initiated to study π -Ricci soliton, which adhere to a specific equation.

$$L_\xi g + 2S + 2lg + 2h\pi \otimes \pi = 0, \qquad (2)$$

and Blaga[4] introduced π-Einstein soliton in 2017, which satisfies the equation

$$L_\xi g + 2S + (2l - scal)g + 2h\pi \otimes \pi = 0, \qquad (3)$$

where $g, S, \xi, scal, \pi$ are pseudo-Riemannian metric, Ricci tensor, a vector field, scalar curvature, a 1-form and *l* and *h* are real constants.

A set of data represented by (g, ξ, l, h), satisfying equations (2) and (3), is termed an π-Ricci soliton and π-Einstein soliton in *M*, respectively. Specifically, when *h* = 0, the pair (g, ξ, l) constitutes a Ricci soliton and Einstein soliton, denoted as steady, shrinking, or expanding depending on whether *l* is zero, negative, or positive, respectively. The study of the kinematic and dynamic nature of relativistic spacetime application in relativity has a physical model of three classes namely: shrinking, steady, and expanding. Such a physical model are similar to solitons of Ricci and Einstein flows.

Now, expressing the Lie derivative $(L_\xi g)(A,T)$ explicitly, we obtain:

$$(L_\xi g)(A,T) = g(\nabla_A \xi, T) + g(A, \nabla_T \xi) \qquad (4)$$

and from (2) and (3) we obtain:

$$S(A,T) = -lg(A,T) - h\pi(A)\pi(T) - \frac{1}{2}[g(\nabla_A \xi, T) + g(A, \nabla_T \xi)], \quad (5)$$

$$S(A,T) = -(l - \frac{scal}{2})g(A,T) - h\pi(A)\pi(T) - \frac{1}{2}[g(\nabla_A \xi, T) + g(A, \nabla_T \xi)], (6)$$

for any $A, T \in \chi(M)$.

In the past two decades, numerous differential geometers have systematically explored symmetry properties utilizing diverse geometric tools such as vector fields[5] curvature tensors[6], and importantly geometric flows in perfect fluid space-time[7,8,9] (briefly say, PFS). Notably, in 2014, Ali and Ahsan[10] initiated an investigation into Ricci solitons and symmetries within PFS. Blaga, further delved into the geometric aspects of PFS, examining Ricci solitons, Einstein solitons, as well as their extensions namely, π-Ricci solitons and π-Einstein solitons within a PFS. In 2019, authors[11] explored the Ricci soliton structure in a PFS where the time-like velocity vector field ξ takes on a torse-forming configuration. Moving into 2022, Siddiqi and De[12] discussed Relativistic PFS and Ricci-Yamabe solitons. Recently, Praveena *et al.*[13,14], described solitons in an almost pseudo symmetric Kähler and Kähler Norden space-time manifold, considering various curvature tensors. Drawing inspiration from these studies, the current paper delves into the geometrical behavior of PFS, particularly focusing on the presence of Ricci and Einstein solitons.

2.0 Preliminaries

The Einstein equation with a cosmological constant for PFS is equivalent to Einstein's equation with dust fluid, and it is expressed as follows:

$$S(A,T) = -(\Lambda - \frac{scal}{2} - k\Omega)g(A,T) + k(\sigma+\Omega)\pi(A)\pi(T), \quad (7)$$

for any $A, T \in \chi(M)$. Where $\Lambda, \sigma, \Omega, k, g$ are the cosmological constant, energy-density, isotropic pressure, gravitational constant, metric tensor of Minkowski spacetime[15] respectively and $\pi(\xi) = -1$ is a 1-form. Also, ξ is velocity vector field such that $g(\xi,\xi) = -1$. If σ = 3p then the perfect fluid space is radiation fluid. Contracting (7) and using $g(\xi,\xi) = -1$, we obtain:

$$scal = k(\sigma - 3\Omega) + 4\Lambda, \quad (8)$$

and from (7) we get

$$S(A,T) = (\Lambda + \frac{k(\sigma-\Omega)}{2})g(A,T) + k(\sigma+\Omega)\pi(A)\pi(T), \quad (9)$$

for any $A, T \in \chi(M)$.

Let's ρ is a torse-forming vector field (briefly say, TFVF) which is of the form

$$\nabla_V \rho = V + A(V)\rho. \quad (10)$$

On PFS admitting TFVF we have the following relations holds:

$$R(V,Y)\rho = \pi(Y)V - \pi(V)Y \quad (11)$$

$$\pi(R(V,Y)Z) = -\pi(Y)g(V,Z) + \pi(V)g(Y,Z), \quad (12)$$

2.1 Solution of Ricci and Einstein Flows in Quasi-Conformal Pseudo Ricci-Symmetric and Quasi-Conformal Flat PFS

The quasi-conformal curvature tensor is defined as[16]

$$\bar{\zeta}(A,T)C = \alpha R(A,T)C + \beta[S(T,C)A - S(A,C)T + g(T,C)QA - g(A,C)QT]$$

$$-\frac{r}{4}\left(\frac{\alpha}{3} + 2\beta\right)[g(T,C)A - g(A,C)T], \quad (13)$$

If the tensors $Q(g,S)$ and $\bar{\zeta}\cdot S$ are linearly dependent then M is called Quasi-conformal pseudo-symmetric curvature conditions and is defined as

$$(\bar{\zeta}(\xi,A)\cdot S)(T,C) = L_S[(\xi \wedge_g A)\cdot S(T,C)]$$

for each of the tangent space, where L_S is some function on U_s, the set

$$U_S = \{D \in M : Q(g,S) \neq 0, at\ \ D\}$$ and

$$(D \wedge_g A)T = g(A,T)D - g(D,T)A.$$

The above condition is equivalent to

$$S(\bar{\zeta}(\xi,A)T,C) + S(T,\bar{\zeta}(\xi,A)C) = L_S\{S((\xi \wedge_g A)T,C) + S(T,(\xi \wedge_g A)C)\} \quad (14)$$

Using equations (9), (11), and (12) in (14) we get

$$\alpha[(\Lambda + \frac{k(\sigma-\Omega)}{2})[g(T,A)\pi(C) - g(T,C)\pi(A)] - k(\sigma+\Omega)[g(T,A)\pi(C)$$

$$+g(A,C)\pi(T) + 2\pi(A)\pi(C)\pi(T)]]\beta[2k(\sigma+\Omega)(\Lambda + \frac{k(\sigma-\Omega)}{2})[-g(A,T)\pi(C)$$

$$-2\pi(A)\pi(C)\pi(T) - \pi(T)g(A,C)] + (k(\sigma+\Omega))^2[g(A,T)\pi(C) + \pi(T)g(A,C)$$

$$+\pi(A)\pi(C)\pi(T)]] + \frac{scal}{4}[\frac{\alpha}{3} + 2\beta]k(\sigma+\Omega)[\pi(T)g(A,C) + \pi(A)\pi(C)\pi(T)]$$

$$= -L_S k(\sigma+\Omega)[g(A,T)\pi(C) + 2\pi(A)\pi(C)\pi(T) + g(A,C)\pi(T)] \quad (15)$$

Let us take $T = \xi$ in preceding equation we gain

$$k(\sigma+\Omega)[\alpha + 2\beta(\Lambda + k\Omega) - \frac{scal}{4}[\frac{\alpha}{3} + 2\beta] - L_S][g(A,C) + \pi(A)\pi(C)]] = 0 \quad (16)$$

for each of the tangent space and from (16) we get

$$\Omega = -\sigma \text{ or } \Omega = \frac{16\alpha(1-\Lambda) + k(\alpha+6\beta)\sigma - 12L_S}{3(k\alpha + 14\beta k)}.$$

Hence we declare the following:

Theorem 1 Let M be a PFS admitting with TFVF ρ if $(\bar{\zeta}(\xi,A)\cdot S)(T,C) = L_S[(\xi \wedge_g A)\cdot S(T,C)]$ *then*

$$\Omega = -\sigma \text{ or } \Omega = \frac{16\alpha(1-\Lambda) + k(\alpha+6\beta)\sigma - 12L_S}{3(k\alpha + 14\beta k)}.$$

We study when TFVF is Quasi-conformaly flat: i.e., $\bar{\zeta}(A,T)C = 0$.

Then equation (13) becomes

$$\alpha R(A,T)C = -\beta[S(T,C)A - S(A,C)T + g(T,C)QA - g(A,C)QT]$$

$$+\frac{scal}{4}\left(\frac{\alpha}{3} + 2\beta\right)[g(T,C)A - g(A,C)T], \quad (17)$$

Contracting A over E_i in (17) then we get

$$S(T,C) = \frac{(\alpha+2\beta)scal}{4(\alpha-2\beta)}g(T,C) \quad (18)$$

From (5) and (18) we get

$$\frac{(\alpha+2\beta)scal}{4(\alpha-2\beta)}g(T,C) = -ag(T,C) - b\pi(T)\pi(C) - \frac{1}{2}[g(\nabla_T\xi,C) + g(T,\nabla_C\xi)] \quad (19)$$

Setting $T = C := Ei$ and using equation (8), we get:

$$4l - h = -\frac{(\alpha+2\beta)}{4(\alpha-2\beta)}(k(\sigma-3\Omega) + 4\Lambda) - div\xi. \quad (20)$$

Writing $X = Y = \xi$ in (19) we obtain:

$$l - h = -\frac{(\alpha+2\beta)}{4(\alpha-2\beta)}(k(\sigma-3\Omega) + 4\Lambda). \quad (21)$$

Solving (20) and (21) we have

$$l = -\frac{(\alpha+2\beta)}{4(\alpha-2\beta)}(k(\sigma-3\Omega)+4\Lambda) - \frac{div\xi}{3} \text{ and } h = \frac{-div\xi}{3}. \quad (22)$$

Thus we state the follwing:

Theorem 2 Let (g,ξ,l,h) be an π-Ricci soliton in a Quasi-conformaly flat PFS manifold then space time is shrinking, steady and expanding according as

$$-\frac{(\alpha+2\beta)}{4(\alpha-2\beta)}(k(\sigma-3\Omega)+4\Lambda) - \frac{div\xi}{3}.$$

From (6) and (18) we get

$$\frac{(\alpha+2\beta)scal}{4(\alpha-2\beta)}g(T,C) = -(l-\frac{scal}{2})g(T,C) - h\pi(T)\pi(C) - \frac{1}{2}[g(\nabla_T\xi,C) + g(T,\nabla_C\xi)], \quad (23)$$

Contracting above equation and using Eq. (8), we obtain:

$$4l - h = \frac{(\alpha-6\beta)}{(\alpha-2\beta)}(k(\sigma-3\Omega)+4\Lambda) - div\xi, \quad (24)$$

Writing $T = C = \xi$ in (23) and using (8) we obtain:

$$l - h = \frac{(\alpha-6\beta)}{4(\alpha-2\beta)}(k(\sigma-3\Omega)+4\Lambda), \quad (25)$$

solving (24) and (25) we get

$$l = -\frac{(\alpha-6\beta)}{4(\alpha-2\beta)}(k(\sigma-3\Omega)+4\Lambda) - \frac{div\xi}{3} \text{ and } h = -\frac{div\xi}{3} \quad (26)$$

Hence we declare the following :

Theorem 3 Let (g,ξ,l,h) be an π-Einstein soliton in a Quasi-conformaly flat PFS manifold then space time is expanding, steady or shrinking according as

$$\frac{(\alpha-6\beta)}{4(\alpha-2\beta)}(k(\sigma-3\Omega)+4\Lambda) + \frac{div\xi}{3}.$$

2.2 Solution of Ricci and Einstein Flows in Pseudo Projective Pseudo Ricci-Symmetric and Pseudo-Projective Flat PFS

The pseudo projective curvature tensor is expressed as follows[17]

$$\bar{P}(A,T)C = \alpha R(A,T)C + \beta[S(T,C)A - S(A,C)T] - \frac{scal}{4}\left(\frac{\alpha}{3}+\beta\right)[g(T,C)A - g(A,C)T]. \quad (27)$$

If the tensors $Q(g,S)$ and $\bar{P}\cdot S$ are linearly dependent then M is called pseudo projective pseudo Ricci-symmetric and is defined as

$$(\bar{P}(\xi,T)\cdot S)(C,D) = L_S[(\xi \wedge_g T)\cdot S(C,D)]$$

for each of the tangent space.

The pseudo projective pseudo-symmetric conditions is equivalent to

$$S(\bar{P}(\xi,T)C,D) + S(C,\bar{P}(\xi,T)D) = L_S\{S((\xi \wedge_g T)C,D) + S(C,(\xi \wedge_g T)D)\} \quad (28)$$

Using equations (9),(11) and (12) in (28) we obtain

$$[L_S - (\alpha - \frac{scal}{12})(\alpha+3\beta)]k(\sigma+\Omega)[g(T,C)\pi(D) + g(T,D)\pi(C) + 2\pi(T)\pi(C)\pi(D)] = 0 \quad (29)$$

Setting $C = \xi$ in the earlier equation we get

$$[(\alpha - \frac{scal}{12})(\alpha+3\beta) - L_S]k(\sigma+\Omega)[g(T,D) + \pi(T)\pi(D)] = 0 \quad (30)$$

for each of the tangent space and we get $\Omega = -\sigma$ or

$$\Omega = \frac{12L_S - (\alpha+3\beta)(12\alpha+4\Lambda+k\sigma)}{3k(\alpha+3\beta)}.$$

Hence we declare the following:

Theorem 4 Let M be a PFS admitting with TFVF Ξ if

$$(\bar{P}(\xi,Y)\cdot S)(Z,W) = L_S[(\xi \wedge_g Y)\cdot S(Z,W)]$$

then $\Omega = -\sigma$ *or* $\Omega = \frac{12L_S - (\alpha+3\beta)(12\alpha+4\Lambda+k\sigma)}{3k(\alpha+3\beta)}$.

We study when PFS is pseudo projective flat: i.e.,

$$\bar{P}(A,T)C = 0.$$

Then equation (27) becomes

$$R(A,T)C = -\frac{\beta}{\alpha}[S(T,C)A - S(A,C)T] + \frac{scal}{4}\left(\frac{\alpha}{3}+\beta\right)[g(T,C)A - g(A,C)T]. \quad (31)$$

Contracting A over E_i then above equation we get

$$S(T,C) = \frac{scal}{4}g(T,C) \quad (32)$$

From (5) and (32) we get

$$\frac{scal}{4}g(T,C) == -lg(T,C) - h\pi(T)\pi(C) - \frac{1}{2}[g(\nabla_T\xi,C) + g(T,\nabla_C\xi)]. \quad (33)$$

Setting $T = C := Ei$ in (33) and using Eq. (8), we get:

$$4l - h = -4\Lambda - k(\sigma-3\Omega) - div\xi. \quad (34)$$

Writing $T = C = \xi$ in (33) we obtain:

$$l - h = -\Lambda - \frac{k}{4}(\sigma-3\Omega). \quad (35)$$

Solving (34) and (35) we have

$$l = -\Lambda - \frac{k}{4}(\sigma-3\Omega) - \frac{div\xi}{3} \text{ and } h = \frac{-div\xi}{3} \quad (36)$$

Therefore, we conclude the following:

Theorem 5 Let (g,ξ,l,h) be an π-Ricci soliton in a pseudo projective flat PFS manifold then space time is expanding, steady or shrinking according as

$$\Lambda - \frac{k}{4}(\sigma - 3\Omega) + \frac{div\xi}{3}.$$

From (6) and (32) we get

$$\frac{scal}{4}g(T,C) = -(l - \frac{scal}{2})g(T,C) - h\pi(T)\pi(C) - \frac{1}{2}[g(\nabla_T\xi,C) + g(T,\nabla_C\xi)],$$

then above equation we can write

$$[l - \frac{scal}{2}]g(T,C) + h\pi(T)\pi(C) + \frac{1}{2}[g(\nabla_T\xi,C) + g(T,\nabla_C\xi)], = 0. \quad (37)$$

Multiplying (37) by ε_{ii} and summing over i for $T = C := Ei$ then using equation (8) , we get:

$$4l - h = 4\Lambda + k(\sigma - 3\Omega) - div\xi, \quad (38)$$

Writing (37) for $T = C = \xi$, we obtain:

$$l - h = \Lambda + \frac{k}{4}(\sigma - 3\Omega), \quad (39)$$

solving (38) and (39) we get

$$l = \Lambda + \frac{k}{4}(\sigma - 3\Omega) - \frac{div\xi}{3} \quad and \quad h = -\frac{div\xi}{3}. \quad (40)$$

Thus, we state the following

Theorem 6 Let $g(\xi, l, h)$ be an π-Einstein soliton in a pseudo projective flat PFS manifold then space time is shrinking, stedy or expanding $\Lambda + \frac{k}{4}(\sigma - 3\Omega) - \frac{div\xi}{3}$.

3.0 Conclusion

Mines in the context of space-time manifold play a pivotal role in scientific endeavors, enabling the study of cosmic phenomena and the unraveling of the universe's mysteries. By strategically placing sensor-equipped mines, scientists can monitor gravitational waves, map the distribution of dark matter, and observe the dynamics of black holes, shedding light on the fundamental workings of the cosmos. The applications of the energy-momentum tensor are widespread, encompassing cosmology and stellar structure, with specific examples found in electromagnetism and scalar field theory. The exploration of the kinematic and dynamic aspects of relativistic space-time within the realm of relativity gives rise to a physical model categorized into three classes: shrinking, steady, and expanding. Notably, this physical model aligns with the characteristics exhibited by Ricci and Einstein flows. Ricci solitons and Einstein solitons emerge as solutions to these flows, and within the context of PFS, their properties are contingent on key factors such as energy density, isotropic pressure, cosmological constant, and gravitational constant. Notably, the cosmological constant can be expressed as a linear combination of energy density, isotropic pressure, and the gravitational constant.

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