Journal of Mines, Metals and Fuels, 71(5): 578-582; 2023. DOI: 10.18311/jmmf/2023/34153

Print ISSN : 0022-2755

Journal of Mines, Metals and Fuels

Contents available at: www.informaticsjournals.com/index.php/jmmf

Generalized Differential Transform Method – Application to Integral Equations of Fractional Order

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Abstract

In this paper, we have used gener-alized differential transform method to solve different types of integral equations of fractional order. All the fractional integrals are written in the Riemann-Liouville sense and fractional derivatives are written in the Caputo sense.

Keywords: *Abel integral equation, Differential transform method, Generalized differential transform method, Caputo sense.*

1.0 Introduction

Abel integral equations are one kind of fractional integral equations. Abel integral equations of fractional order $^1/_2$ of first kind are of two types given by

$$\int_a^x \frac{\phi(t)}{(x-t)^{\frac{1}{2}}}dt = f(x), \quad x > a \quad (f(a) = 0) \tag{1.1}$$

and

$$\int_x^b \frac{\phi(t)}{(t-x)^{\frac{1}{2}}}dt = f(x), \quad a < x < b \quad (f(b) = 0) \tag{1.2}$$

A generalization of (1.1) and (1.2) is

$$\int_a^x \frac{\phi(t)}{(x-t)^{\alpha}}dt = f(x), \quad x > a \quad (f(a) = 0) \tag{1.3}$$

and

$$\int_x^b \frac{\phi(t)}{(t-x)^{\alpha}}dt = f(x), \quad a < x < b \quad (f(b) = 0) \tag{1.4}$$

Here α is a constant and lies between 0 and 1, $f(x)$ is the known function and $\phi(x)$ is the unknown function to be determined. Exact solutions of (1.3) and (1.4) are given by

$$\phi(x) = \frac{\sin \pi\alpha}{\pi}\frac{d}{dx}\left[\int_a^x \frac{f(t)}{(x-t)^{1-\alpha}}dt\right] \tag{1.5}$$

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And

$$\phi(x) = \frac{\sin \pi\alpha}{\pi} \frac{d}{dx}\left[\int_x^b \frac{f(t)}{(t-x)^{1-\alpha}} dt\right] \tag{1.6}$$

respectively,

Yousefi[1] used Legendre wavelets for the numerical solution of Abel integral equation. Liu and Tao[2] used mechanical quadrature methods, Dixit et al[3] used Bernstein operational matrix and Pandey et al[4] used collocation method to solve the generalized Abel integral equations.

In this paper we will use the following result

$$\int_0^x \frac{t^{\beta k}}{(x-t)^{\alpha}} dt = \frac{\Gamma(1+\beta k)\Gamma(1-\alpha)}{\Gamma(2+\beta k-\alpha)} x^{1+\beta k-\alpha}, \tag{1.7}$$

$0 < \propto < 1.$

2.0 Analysis of Generalized Differential Transform Method

Odibat et al [6] obtained generalized differential transform method to solve differential equations of fractional order numerically.

The generalized differential transform is given as follows:

$$F_\alpha(k) = \frac{1}{\Gamma(\alpha k+1)}\left[\left(D_{x_0}^{\alpha}\right)^k f(x)\right]_{x=x_0} \tag{2.1}$$

and $f(x) = \sum_{k=0}^{\infty} F_\alpha(k)(x-x_0)^{\alpha k}$. (2.2)

In particular application, the function $f(x)$ can be approximated by the finite series

$$f(x) = \sum_{k=0}^{N} F_\alpha(k)(x-x_0)^{\alpha k}. \tag{2.3}$$

Substitution of equation (2.1) in (2.2) yields

$$f(x) = \sum_{k=0}^{\infty} \frac{(x-x_0)^{\alpha k}}{\Gamma(\alpha k+1)}\left[\left(D_{x_0}^{\alpha}\right)^k f(x)\right]_{x=x_0} \tag{2.4}$$

Properties

P1: If $f(x) = g(x) \pm h(x)$, then $F_\alpha(k) = G_\alpha(k) \pm H_\alpha(k)$

P2 : If $f(x) = ag(x)$, then $F_\alpha(k) = aG_\alpha(k)$.

P3 : If $f(x) = g(x)h(x)$, then $F_\alpha(k) = \sum_{l=0}^{k} G_\alpha(l)H_\alpha(k-l)$.

P4 : If $f(x) = D_{x_0}^{\alpha} g(x)$, then $F_\alpha(k) = \frac{\Gamma[\alpha(k+1)+1]}{\Gamma[\alpha k+1]} G_\alpha(k+1)$.

P5 : If $f(x) = (x-x_0)^{n\alpha}$, then $F_\alpha(k) = \delta(k-n)$

$$= \begin{cases} 1 & if\ k = n \\ 0 & if\ k \neq n. \end{cases}$$

3.0 Method of Solution

We will solve the integral equations (1.3) and (1.4) by using generalized differential transform method (GDTM).

Let us take the following transformations:

$t - a = z, x - a = y, \phi(z+a) = \phi_1(z), f(y+a) = f_1(y)$, equation (1.3) becomes

$$\int_0^y \frac{\phi_1(z)}{(y-z)^{\alpha}} dz = f_1(y), y > 0, f_1(0) = 0, \tag{3.1}$$

Let $\phi_1(x) = \sum_{k=0}^{\infty} \phi_{1-\alpha}(k)x^{k(1-\alpha)}$ (3.2)

And $f_1(x) = \sum_{k=0}^{\infty} F_{1-\alpha}(k)x^{k(1-\alpha)}$ (3.3)

Where $\phi_{1-\alpha}(k)$ and $F_{1-\alpha}(k)$ are the generalized DT of $F_1(x)$ and $\phi_1(x)$ respectively.

By using the equation (3.2) and the result (1.7) in the equation (3.1) we get

$$\sum_{k=0}^{\infty} \phi_{1-\alpha}(k)\left[\frac{\Gamma[1+(1-\alpha)k]\Gamma(1-\alpha)}{\Gamma[2+(1-\alpha)k-\alpha]}\right] y^{(1-\alpha)(k+1)} = f_1(y) \tag{3.4}$$

Use the properties P2 and P5, (3.4) becomes

$$\sum_{k=0}^{\infty} \phi_{1-\alpha}(k)\left[\frac{\Gamma[1+(1-\alpha)k]\Gamma(1-\alpha)}{\Gamma[2+(1-\alpha)k-\alpha]}\right] \delta[m-(k+1)] = F_{1-\alpha}(m) \tag{3.5}$$

This gives

$$\phi_{1-\alpha}(m) = F_{1-\alpha}[m+1]\frac{\Gamma[2+(1-\alpha)m-\alpha]}{\Gamma[1+(1-\alpha)m-\alpha]\Gamma[1-\alpha]}$$

Which can be written as

$$\phi_{1-\alpha}(m) = \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)} \sum_{k=0}^{\infty} F_{1-\alpha}[k] \frac{\Gamma[1+(1-\alpha)k]\Gamma[\alpha]}{\Gamma[1+(1-\alpha)k+\alpha]} [(1-\alpha)k+\alpha]\delta[m-(k-1)]. \tag{3.6}$$

By using inverse GDT we get

$$\phi_1(y) = \frac{\sin \alpha\pi}{\pi} \sum_{k=0}^{\infty} F_{1-\alpha}(k) \frac{\Gamma[1+(1-\alpha)k]\Gamma[\alpha]}{\Gamma[1+(1-\alpha)k+\alpha]} [(1-\alpha)k+\alpha] y^{(1-\alpha)(k-1)}.$$

$$= \frac{\sin \alpha\pi}{\pi} \frac{d}{dy}\left[\sum_{k=0}^{\infty} F_{1-\alpha}(k) \frac{\Gamma[1+(1-\alpha)k]\Gamma[\alpha]}{\Gamma[1+(1-\alpha)k+\alpha]} y^{(1-\alpha)k+\alpha}\right]$$

$$= \frac{\sin \alpha\pi}{\pi} \frac{d}{dy}\left[\sum_{k=0}^{\infty} F_{1-\alpha}(k) \int_0^y \frac{z^{(1-\alpha)k}}{(y-z)^{1-\alpha}} dz\right]$$

$$= \frac{\sin \alpha\pi}{\pi} \frac{d}{dy}\left[\int_0^y \frac{\sum_{k=0}^{\infty} F_{1-\alpha}(k) z^{(1-\alpha)k}}{(y-z)^{1-\alpha}} dz\right]$$

$$= \frac{\sin \alpha\pi}{\pi} \frac{d}{dy}\left[\int_0^y \frac{f_1(z)}{(y-z)^{1-\alpha}} dz\right]$$

Nothing that $\phi_1(y) = \phi(y+a), f_1(z) = f(z+a)$ and back substitution of $z=t-a$, $y=x-a$ in (3.7) we get the solution of the integral equation (1.3) as given by (1.5).

Now by using the following transformations

$b-t=z, b-x=y, \phi(b-z)=\phi_2(z), f(b-y)=f_2(y)$ the equation (1.4) becomes

$$\int_0^y \frac{\phi_2(z)}{(y-z)^{\alpha}} dz = f_2(y), y>0, f_2(0)=0. \quad (3.8)$$

Similarly we can get,

$$\phi_2(y) = \frac{\sin\alpha\pi}{\pi}\frac{d}{dy}\left[\int_0^y \frac{f_2(z)}{(y-z)^{1-\alpha}} dz\right] \quad (3.9)$$

And by using $\phi_2(y) = \phi(b-y), f_2(z) = f(b-z)$ and back substitution of $z=b-t$, $y=b-x$ into (3.9) we get the solution of the integral equation (1.4) as given by (1.6).

4.0 Illustrative Examples

Example 1. Here we have consider the integral equation of the first kind given by

$$\int_0^x \frac{\phi(t)}{(x-t)^{\frac{1}{2}}} dt = e^x - 1, \quad (4.1)$$

with the exact solution $\phi(x) = \frac{e^x erf(\sqrt{x})}{\sqrt{\pi}}$.

The integral equation (4.1) is fractional integral equation of order $^1/_2$.

Given $f(x) = e^x - 1 = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$$F_{\frac{1}{2}}(m) = \delta(m-2) + \frac{1}{2!}\delta(m-4) + \frac{1}{3!}\delta(m-6) + \cdots \ldots \ldots \ldots$$

Therefore,

$$F_{\frac{1}{2}}(m) = \begin{cases} 1 & m=2, \\ \frac{1}{2!} & m=4, \\ \frac{1}{3!} & m=6, \\ \frac{1}{4!} & m=8 \end{cases}$$

... and so on.

Therefore by using (3.6), the GDT of equation (4.1) is

$$\sum_{k=0}^{\infty} \phi_{\frac{1}{2}}(k) \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(1+\frac{k}{2}\right)}{\Gamma\left(\frac{3}{2}+\frac{k}{2}\right)} \delta[m-(1+k)] = F_{\frac{1}{2}}(m)$$

So that

$$\phi_{\frac{1}{2}}(m-1) \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(1+\frac{m-1}{2}\right)}{\Gamma\left(\frac{3}{2}+\frac{m-1}{2}\right)} = F_{\frac{1}{2}}(m)$$

Putting m = 1,2,3,…… one finds

$$\phi_{\frac{1}{2}}(k) = \begin{cases} \frac{2}{\pi} & k=1, \\ \frac{4}{3\pi} & k=3, \\ \frac{8}{15\pi} & k=5, \end{cases}$$

…….and so on.

Now, from inverse GDT, we get

$$\phi(x) = \sum_{k=0}^{\infty} \phi_{\frac{1}{2}}(k) x^{\frac{k}{2}}$$

$$= \phi_{\frac{1}{2}}(1)x^{\frac{1}{2}} + \phi_{\frac{1}{2}}(3)x^{\frac{3}{2}} + \phi_{\frac{1}{2}}(5)x^{\frac{5}{2}} + \cdots$$

$$= \frac{2}{\pi}x^{\frac{1}{2}} + \frac{4}{3\pi}x^{\frac{3}{2}} + \frac{8}{15\pi}x^{\frac{5}{2}} + \cdots = \frac{e^x erf(\sqrt{x})}{\sqrt{\pi}}.$$

Example 2. In this example we consider the integral equation of the second kind given by

$$\phi(x) + \int_0^x \frac{\phi(t)}{(x-t)^{\frac{1}{2}}} dt = x^2 + \frac{16}{15}x^{\frac{5}{2}}, \quad (4.2)$$

With exact solution $\phi(x) = x^2$.

Here the order of the fractional integral equation (4.2) is $^1/_2$.

Given $f(x) = x^2 + \frac{16}{15}x^{\frac{5}{2}}$

So, $F_{\frac{1}{2}}(m) = \delta(m-4) + \frac{16}{15}\delta(m-5)$

Therefore,

$$F_{\frac{1}{2}}(m) = \begin{cases} 1 & m=4, \\ \frac{16}{15} & m=5, \\ 0, & otherwise. \end{cases}$$

By using (3.6), the GDT of equation (4.2) is obtained as

$$\phi_{\frac{1}{2}}(m) + \sum_{k=0}^{\infty} \phi_{\frac{1}{2}}(k) \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(1+\frac{k}{2}\right)}{\Gamma\left(\frac{3}{2}+\frac{k}{2}\right)} \delta[m-(1+k)]$$

$$= F_{\frac{1}{2}}(m)$$

So that,

$$\phi_{\frac{1}{2}}(m) + \phi_{\frac{1}{2}}(m-1) \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(1+\frac{m-1}{2}\right)}{\Gamma\left(\frac{3}{2}+\frac{m-1}{2}\right)} = F_{\frac{1}{2}}(m).$$

For m = 0, 1, 2, 3, ... we get the system of linear equations as

$$\phi_{\frac{1}{2}}(0) = 0,$$

$$\phi_{\frac{1}{2}}(1) + 2\phi_{\frac{1}{2}}(0) = 0,$$

$$\phi_{\frac{1}{2}}(2) + \frac{\pi}{2}\phi_{\frac{1}{2}}(1) = 0,$$

$$\phi_{\frac{1}{2}}(3) + \frac{4}{3}\phi_{\frac{1}{2}}(2) = 0,$$

$$\phi_{\frac{1}{2}}(4) + \frac{3\pi}{8}\phi_{\frac{1}{2}}(3) = 1,$$

$$\phi_{\frac{1}{2}}(5) + \frac{16}{15}\phi_{\frac{1}{2}}(4) = \frac{16}{15}$$

So that

$$\phi_{\frac{1}{2}}(k) = \begin{cases} 1 & k = 4 \\ 0\ , & otherwise. \end{cases}$$

Now, from inverse generalized differential transform, one obtains

$$\phi(x) = \sum_{k=0}^{\infty} \phi_{\frac{1}{2}}(k) x^{\frac{k}{2}}$$

$$= \phi_{\frac{1}{2}}(4)x^2$$

$$= x^2.$$

Example 3. Here we have taken the integral equation of the second kind given by

$$\phi(x) + \int_0^x \frac{\phi(t)}{(x-t)^{\frac{1}{2}}} dt = 2\sqrt{x}, \tag{4.3}$$

with exact solution $\phi(x) = 1 - e^{\pi x} erfc\left(\sqrt{\pi x}\right)$

where $erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-u^2}\, du.$

Here $f(x) = 2\sqrt{x}$, therefore

$F_{\frac{1}{2}}(m) = 2\delta(m-1).$

Therefore,

$$F_{\frac{1}{2}}(m) = \begin{cases} 2 & ,m = 1 \\ o & ,otherwise \end{cases}$$

By using the equation (3.6), the GDT of equation (4.3) is

$$\phi_{\frac{1}{2}}(m) + \sum_{k=0}^{\infty} \phi_{\frac{1}{2}}(k) \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(1+\frac{k}{2}\right)}{\Gamma\left(\frac{3}{2}+\frac{k}{2}\right)} \delta[m-(1+k)] = F_{\frac{1}{2}}(m),$$

So that,

$$\phi_{\frac{1}{2}}(m) + \phi_{\frac{1}{2}}(m-1) \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(1+\frac{m-1}{2}\right)}{\Gamma\left(\frac{3}{2}+\frac{m-1}{2}\right)} = F_{\frac{1}{2}}(m).$$

For m = 0, 1, 2, 3, ... we get the system of linear equations as

$$\phi_{\frac{1}{2}}(0) = 0,$$

$$\phi_{\frac{1}{2}}(1) + 2\phi_{\frac{1}{2}}(0) = 2,$$

$$\phi_{\frac{1}{2}}(2) + \frac{\pi}{2}\phi_{\frac{1}{2}}(1) = 0,$$

$$\phi_{\frac{1}{2}}(3) + \frac{4}{3}\phi_{\frac{1}{2}}(2) = 0,$$

$$\phi_{\frac{1}{2}}(4) + \frac{3\pi}{8}\phi_{\frac{1}{2}}(3) = 0,,$$

and so on

Which has the following solution

$$\phi_{\frac{1}{2}}(k) = \begin{cases} 0 & k = 0 \\ 2 & k = 1 \\ -\pi & k = 2 \end{cases}$$

And so on

Now, from inverse GDT, we get

$$\phi(x) = \sum_{k=0}^{\infty} \phi_{\frac{1}{2}}(k) x^{\frac{k}{2}}$$

$$= \phi_{\frac{1}{2}}(1)x^{\frac{1}{2}} + \phi_{\frac{1}{2}}(2)x^{\frac{2}{2}} + \phi_{\frac{1}{2}}(3)x^{\frac{3}{2}} + \phi_{\frac{1}{2}}(4)x^{\frac{4}{2}} + \cdots$$

$$= 2\sqrt{x} - \pi x + \frac{4\pi x^{\frac{3}{2}}}{3} - \frac{\pi^2 x^2}{2} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(\pi x)^{\frac{n}{2}}}{\Gamma\left(1+\frac{n}{2}\right)}$$

$$= 1 - E_{\frac{1}{2}}\left(-\sqrt{\pi x}\right)$$

$$= 1 - e^{\pi x} erfc\left(\sqrt{\pi x}\right),$$

Which is the exact solution and $F_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)}$ is the Mittag-Leffler function.

5.0 Conclusion

In the present work, we have used generalized differential transform method to obtain numerical solution of the singular integral equation of fractional order. We have given some examples using the proposed method. We can see that the proposed mechanism works well, very easy to apply and has good accuracy.

6.0 References

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