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Solution of Lane-Emden Type Differential Equations by Using Differential Transform Method

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Abstract

The temperature in a self-gravitating star has been widely described in astrophysics by the Lane-Emden differential equation. To solve the non-linear equations, the Lane-Emden differential equation is used as a prototype for testing new mathematical and numerical techniques. The problems of singular initial value of Lane-Emden type was solved by the Differential Transform Method in this study. This method is applicable to solve various linear and nonlinear problems which decrease size of computational work. Here, we introduced some numerical problems to describe the high accuracy and efficiency of our proposed method.

Keywords: *Differential Transform Method (DTM), Lane-Emden Differential Equations.*

1.0 Introduction

The main problem from the beginning of stellar astrophysics was to find out the stellar structures. In astrophysics, when investigation was continued about stellar structures then Lane-Emden equation gave a remarkable result to decrease the density and mass of a star. The history of a spherical cloud of gas and isothermal gas sphere are described by the Lane-Emden equations. Gaseous star's density profile was briefly pointed out by this equation. Dimensionless form of Poisson's equation for the gravitational potential of a self-gravitating, spherically symmetric polytropic fluid is also described by Lane-Emden equation.

General form of second order Lane-Emden equation is given by:

$$\frac{d^2\phi}{dx^2} + \frac{C}{x}\frac{d\phi}{dx} + f(x,\phi) = g(x),\ 0 \le x \le 1, c \ge 0, \quad ...(1.1)$$

With supplementary conditions $\phi(0) = A$ and $\phi'(0) = B$, Here A, B are constants and $f(x, \phi)$ is a continuous real valued function of x and ϕ also $g(x) \in C[0,1]$.

Lane-Emden equation (Standard form):

$$c = 2, f(x,\phi) = \phi^m, \phi(0) = 0, g(x) = 0.$$

Then the equation (1.1) becomes

$$\frac{d^2\phi}{dx^2} + \frac{2}{x}\frac{d\phi}{dx} + \phi^m = 0,$$

With initial conditions $\phi(0) = 1$ and $\phi'(0) = 0$.

Some phenomena in mathematics and astrophysics have been modelled by using the Lane-Emden equation. J.H. Lane (1870) first describes the Lane-Emden equation[1]. Then R. Emden (1907) gave an analytical idea about it. On the other hand a multi integral method to investigate the non-linear problem was used by El-Sayed[12]. Various methods such as Adomain Decomposition method (ADM) was introduced by Wazwaz[4], pointed a effective algorithm which was used for solving differential equations of Lane-Emden type. To calculate the initial value problems, Legendre Wavelets method has been applied independently by Yousefi[13]. Chowdhari et al.[3] gave an unique idea about solutions of singular IVP's of Lane-Emden type by Homotopy Perturbation method (HPM). Taylor's series was given by Hasan and Jhu[8]

and convergent series form was also made to solve the problems of singular initial value. Singh et al.[6] introduced a analytical algorithm of Lane-Emden type equation arising in astrophysics using modified Homotopy Analysis Method (HAM). Parand et al.[10] briefly described the Unsupervised combined artificial neural network method. Mall and Chakraverty[7] gave the Chebyshev neural network based model for solving Lane-Emden type equations. Homotopy Analysis Method (HAM) helped to solve the singular IVP's of Lane-Emden type which was introduced by Al-Hayani et.al[5]. Boundary Value Problem of Lane-Emden type was solved by stochastic numerical computing technique with artificial neural network which was analyzed by Ahmad et.al[11]. The Lane-Emden equation was also solved by Jaiswal and Yadav[9] with the help of coupling method of Laplace transform and Chebyshev wavelets.

In present study, at first, linear and nonlinear Lane-Emden's equations were solved by a effective numerical technique differential transform method. To test the validity and efficiency of this method in comparison with other existing methods, some special problems were solved. By using this method we got an approximate solution which pointed out its uniqueness on the other existing numerical results.

2.0 Differential Transform Method

The differential transform of the function $\phi(x)$ is defined as

$$\Phi(k) = \frac{1}{k!}\left[\frac{d^k\varphi}{dx^k}\right]_{x=x_0},$$

And the inverse differential transformation of $\varphi(x)$ is defined as

$$\varphi(x) = \sum_{k=0}^{\infty} \phi(k)(x-x_0)^k$$

In real applications, function $\phi(x)$ is expressed by a finite series and can be written as

$$\varphi(x) = \sum_{k=0}^{\infty} \phi(k)(x-x_0)^k$$

Where

$$\sum_{k=0}^{n} \phi(k)(x-x_0)^k$$

is negligible small.

Theorem 1: If $\varphi(x) = y_1(x) + y_2(x)$, then
$\phi(k) = Y_1(k) + Y_2(K)$.

Theorem 2: If $(x) = ay(x)$, then
$\phi(k) = aY(k)$, where a is a constant.

Theorem 3: If $(x) = \frac{d^m y(x)}{dx^m}$, then

$\phi(k) = \frac{(m+k)!}{k!} Y(k+m)$.

Theorem 4: If $(x) = y_1(x)y_2(x)$, then

$$\phi(k) = \sum_{k_1=0}^{k} Y_1(k_1)Y_2(k-k_1)$$

Theorem 5: If $\varphi(x) = x^n$, then
$\phi(k) = \delta(k-n)$,

$$\delta(k-n) = \begin{cases} 1, & k = n \\ 0, & k \neq n \end{cases}$$

Example 1:

The Lane-Emden equation with $f(x, \phi) = -2(2x^2+3)\phi$, a second order Homogeneous linear differential equation

$$\frac{d^2\phi}{dx^2} + \frac{2}{x}\frac{d\phi}{dx} - 2(2x^2+3)\phi = 0$$

with the initial conditions $\varphi(0) = 1$ and $\phi'(0) = 0$.

Given

$$\frac{d^2\phi}{dx^2} + \frac{2}{x}\frac{d\phi}{dx} - 2(2x^2+3)\phi = 0, x \geq 0,$$

Multiplying both sides by x

$$x\frac{d^2\phi}{dx^2} + x\frac{2}{x}\frac{d\phi}{dx} - 4x^3\phi - 6x\phi = 0$$

By using DTM we have,

$$x\frac{d^2\phi}{dx^2} = \sum_{k_1=0}^{k} Y_1(k_1)\,Y_2(k-k_1)$$

$$= \sum_{k_1=0}^{k} \delta(k_1-1)\frac{(k-k_1+2)!}{(k-k_1)!}\phi(k-k_1+2)$$

$$= \frac{(k+1)!}{(k-1)!}\phi(k+1)$$

$$= k(k+1)\phi(k+1)$$

$$2\frac{d\phi}{dx} = 2\frac{(k+1)!}{k!}\phi(k+1)$$

$$= 2(k+1)\phi(k+1)$$

$$4x^3\phi = 4\sum_{l=0}^{k} Y_1(l)Y_1(k-l)$$

$$= 4\sum_{l=0}^{k} \delta(l-3)\phi(k-l)$$

and

$$6x\phi = 6\sum_{l=0}^{k} Y_1(l)Y_1(k-l)$$

$$= 6\sum_{l=0}^{k} \delta(l-1)\phi(k-l)$$

Therefore

$$\phi(k+1) = \frac{1}{(k+1)(k+2)}\left[4\sum_{l=0}^{k}\delta(l-3)\phi(k-l) + 6\sum_{l=0}^{k}\delta(l-1)\phi(k-l)\right]$$

Solving the system of linear equations and using the initial conditions we get

$\phi(0) = 1$,

$\phi(2) = 1$,

$\phi(4) = \frac{1}{2!}$ ... and

$\phi(1) = \phi(3) = \phi(5) = \phi(7) = \phi(9) = \ldots = 0$

Thus from the inverse differential transform we have,

$$\varphi(x) = \sum_{k=0}^{n}\phi(k)x^k$$

$$= \phi(0)x^0 + \phi(1)x^1 + \phi(2)x^2 + \phi(3)x^3 + \phi(4)x^4 + \cdots$$

$$= 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots$$

$$= e^{x^2}$$

Comparison of exact solution with DTM solution is given in Fig.1.

Example 2:

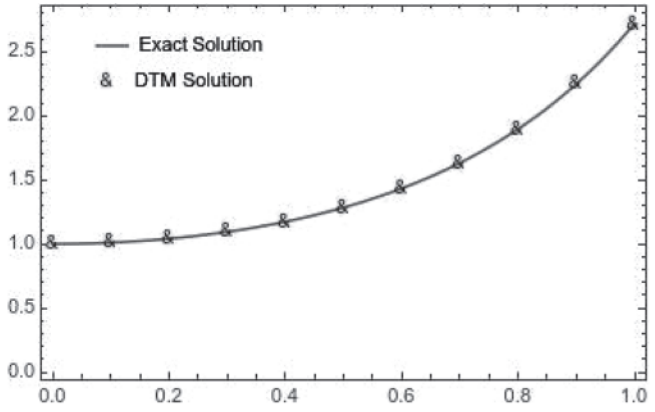


Figure 1: Plot the comparison of exact solution with DTM solution (Example 1)

The Lane-Emden equation with $f(\phi) = \phi^5$, a non-linear homogeneous second order differential equation:

$$\frac{d^2\varphi}{dx^2} + \frac{2}{x}\frac{d\phi}{dx} + \phi^5 = 0, \text{ x} \geq 0$$

With the given initial condition $\phi(0) = 1$ and $\phi'(0) = 0$

By using DTM we have,

$$\sum_{l=0}^{k}\delta(l-1)(k+2-l)\,\Phi(k+2-l) + 2(k+1)\Phi(k+1) + \sum_{r=0}^{k}\sum_{s=0}^{k-r}\sum_{m=0}^{k-r-s}\sum_{p=0}^{k-r-s-m}\sum_{q=0}^{k-r-s-m-p}\delta(r-1)\Phi(s)\Phi(m)\Phi(p)\Phi(q)\Phi(k-r-s-m-p-q) = 0.$$

Solving the system of linear equations and using the initial conditions we get,

$$\phi(2) = -\frac{1}{6}, (4) = \frac{1}{24}, \phi(6) = -\frac{5}{432}, \phi(8) = \frac{35}{10368}, \phi(10) = -\frac{7}{6912}, \ldots \text{ and}$$

$$\phi(1) = \phi(3) = \phi(5) = \phi(7) \ldots = 0.$$

Thus from the inverse differential transform we have,

$$\phi(x) = 1 - \frac{x^2}{6} + \frac{x^4}{24} - \frac{5x^6}{432} + \frac{35x^8}{10368} - \frac{7x^{10}}{6912} + \cdots$$

And thus the solution is in the form

$$\phi(x) = \left(1 + \frac{x^2}{3}\right)^{-\frac{1}{2}}.$$

Comparison of exact solution with DTM solution is given ı Fig.2.

Example 3:

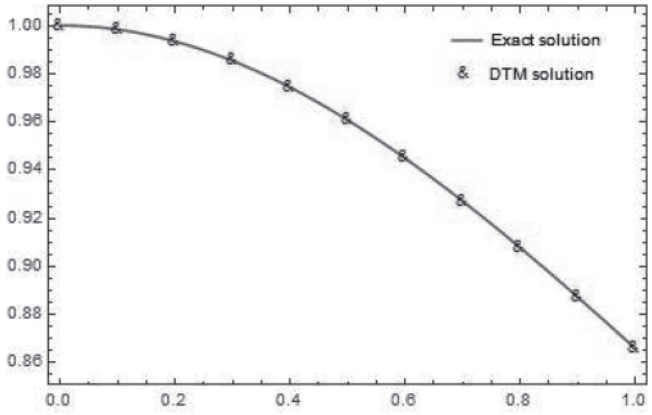


Figure 2: Plot the comparison of exact solution with DTM solution (Example 2)

The Lane-Emden equation of a non-linear homogeneous second order differential equation:

$$\frac{d^2\phi}{dx^2} + \frac{8}{x}\frac{d\phi}{dx} + x\phi(x)$$
$$= x^5 - x^4 + 44x^2 - 30x,\ 0 < x < \infty$$

With the given initial condition $\varphi(0)=0=\phi'(0)$

Given

$$\frac{d^2\phi}{dx^2} + \frac{8}{x}\frac{d\phi}{dx} + x\phi(x)$$
$$= x^5 - x^4 + 44x^2 - 30x, \quad 0 < x < \infty$$

Multiplying both sides by x and using DTM we get,

$$x\frac{d^2\phi}{dx^2} = \sum_{k_1=0}^{k} Y_1(k_1)\, Y_2(k-k_1)$$

$$= \sum_{k_1=0}^{k} \delta(k_1-1)\frac{(k-k_1+2)!}{(k-k_1)!}\phi(k-k_1+2)$$

$$= \frac{(k-k_1+2)!}{(k-1)!}\phi(k-1+2)$$

$$= \frac{(k+1)!}{(k-1)!}\phi(k+1)$$

$$= k(k+1)\phi(k+1)$$

$$8\frac{d\phi}{dx} = 8\frac{(k+1)!}{k!}\phi(k+1)$$

$$= 8(k+1)\phi(k+1),$$

$$x^2\phi = \sum_{l=0}^{k} Y_1(l)Y_1(k-l)$$

$$= \sum_{l=0}^{k} \delta(l-2)\phi(k-l)$$

Therefore

$$\phi(k+1) = \frac{1}{(k+1)(k+8)}\left[\delta(k-6) - \delta(k-5) + 44\delta(k-3) - 30\delta(k-2) - \sum_{l=0}^{k} \delta(l-2)\phi(k-l)\right]$$

Solving the system of linear equations and using the initial conditions we get

$\phi(0)=0$,
$\phi(1)=0$
$\phi(2)=0$,
$\phi(3)=-1$, $\phi(4)=1$, $\phi(5)=0$, ...

Thus using equation we have

$$\varphi(x) = \sum_{k=0}^{n} \phi(k)(x-x_0)^k$$

$$\varphi(x) = \sum_{k=0}^{n} \phi(k)x^k$$

$$= \phi(0)x^0 + \phi(1)x^1 + \phi(2)x^2 + \phi(3)x^3 + \phi(4)x^4 + \ldots$$

$$= -x^3 + x^4$$

And thus the solution in the form

$\phi(x) = -x^3 + x^4$.

Comparison of exact solution with DTM solution is given in Fig.3.

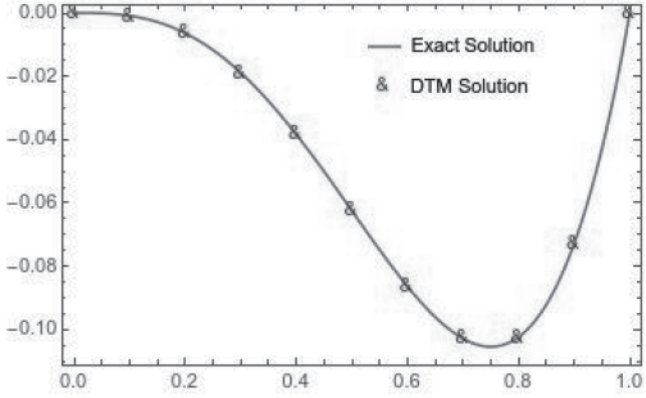


Fig. 3 Plot the comparison of exact solution with DTM solution (Example 3)

3.0 Conclusion

Lane-Emden type equations are numerically solved successfully by applying DTM and Singular initial value problem of Lane-Emden type was also effectively solved by the DTM which is a reliable tool. The DTM minimize the computational difficulties of all the calculations and it can be made easy by simple manipulations. Some problems were solved by using this method and it gave remarkable results.

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