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A Refined Shear Deformation Theory to Analyse a Deep Cantilever Beam Carrying Uniformly Distributed Load for Different Aspect Ratios

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Abstract

This study presents the analysis of a thick cantilever beam subjected to a uniformly distributed load using a seventh-order shear deformation theory. The proposed beam theory incorporates the influence of depth, accounting for shear deformation effects through the displacement field along the thickness coordinate. The displacement field considers both transverse shear and normal deformation effects, with terms expanded up to the seventh order for an accurate prediction of the beam's bending behaviour. Constitutive relations are employed to calculate transverse shear stresses, ensuring zero shear stress on the beam's upper and lower surfaces, thereby eliminating the need for a shear correction coefficient. The governing equations and boundary conditions are derived using the principle of virtual work. An isotropic, homogeneous cantilever beam under a uniformly distributed load is analyzed for varying aspect ratios to demonstrate the efficacy of the seventh-order shear deformation theory. Expressions for transverse displacements, normal displacements, transverse shear stresses, and normal bending stresses are derived and presented through graphical and tabular representations. The results are compared with those obtained from established theories, including the Euler-Bernoulli theory, Timoshenko theory, and other higher-order shear deformation theories, to validate the accuracy and effectiveness of the proposed model. The seventh-order shear deformation theory provides accurate predictions of bending behaviour, eliminating the need for a shear correction coefficient. It shows improved agreement with established theories, and intense beams under varying aspect ratios.

Major Findings: *The seventh-order shear deformation theory effectively predicts bending in deep cantilever beams, aligns well with higher-order theories, and eliminates the need for a shear correction coefficient.*

Keywords: *Axial Stress, Deep Beam, Displacement Field, Equilibrium Equation, Shear Deformation, Shear Stress, Virtual Work Principle*

1.0 Introduction

Shear deformation theory is the crucial component of structural analysis for deep or thick beams and plates where one cannot ignore the shear deformation effect. Classical beam theory like the Euler-Bernoulli theory of beam[1-4] assumes that plane sections, normal to the neutral axis before bending will continue to be normal to it following the bending condition. The transverse normal as well as transverse-shear strain are also zero. The theory ignores Shear Deformation's (SD) effect, consequently suitable for thin beams and not well-suited to deep or thick beams. The shortfall in Euler's theory is eliminated in the first-order theory. Issues with shear and bending

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forces in beams were addressed, significantly contributing to the theoretical framework[5]. The pioneering work in incorporating rotatory inertia as well as shear flexibility as advanced dynamic effects observed in beam theory can be attributed to Rankine[6] and Bresse[7]. Timoshenko established the much stronger effect of transverse shear on prismatic bar transverse vibration response than rotatory inertia, Nevertheless, this theory is known as Timoshenko's theory of beam or 1st order SD theory[8]. According to Timoshenko beam theory, the distribution of transverse shear strain remains uniform over the beam's thickness. Consequently, the shear correction coefficient is necessary to adjust the strain energy of deformation. For the Timoshenko beam theory, the first formulation of exact boundary conditions was achieved, followed by investigations into techniques for reducing three-dimensional issues in elastic body mechanics to develop complex theories[9-11]. Detailed exploration of the Timoshenko theory was conducted for thick beams, addressing fundamental two- and three-dimensional elasticity theory problems[12-14]. A comprehensive analysis identified the shear correction coefficient for various beam cross-sections, with subsequent improvements provided for its equation[15,16].

The limitations of ETB and FSDT are addressed in higher-order theories. An SD theory incorporating a third-order function in the depth or thickness coordinate was developed, satisfying the no shear-stress boundary condition on the beam's top and bottom faces, thus eliminating the need for a shear correction coefficient[17-20]. Higher-order SDT was formulated for rectangular deep beams and later extended to orthotropic beams[21-24]. Advanced beam theories yielding sixth-order differential equations were proposed based on stress assumptions and kinematic frameworks. Parabolic SD theories and further developments accounted for transverse-normal strain and transverse-shear effects on isotropic beam bending[25-29]. Finite element models utilizing higher-order SDT were introduced for rectangular beams, incorporating transverse-shear and normal-strain effects[30-32]. Additional theories, including hyperbolic SDT and a fifth-order SDT for static loading and bending analysis of isotropic plates, expanded the theoretical framework[33-35]. A comprehensive review of advanced SD theories further highlights their evolution and applications.

According to the literature review, which revealed the research gap, for beams that experience shear loads at both their top and bottom, unconstrained advanced theories are rarely available in the literature. Also, plenty of research studies have been reported in the literature up to the 5th order, but it is observed that the analytical solutions for thick beams subjected to various loads using 7th-order shear deformation theory are not available. In the current paper order refined shear deformation theory is presented. The governing equations are formulated from the virtual work's principles. Investigation is carried out for the analysis of deep cantilever beams carrying uniformly distributed loads. The findings based on the present theory are compared with the theories documented in the literature.

2.0 Theoretical Formulation

A rectangular, isotropic and homogeneous beam is taken for the theoretical formulation.

$$\left[0 \le x \le L\right]; \left[-b/2 \le y \le b/2\right]; \left[-h/2 \le z \le h/2\right] \quad (1)$$

L - beam length along the 'x' axis, b - width along the 'y' axis, h - thickness along the 'z' axis

x, y, z - coordinates.

2.1 Displacement Field

The displacement field used to present the refined beam theory is outlined as follows.

$$u(x,z) = z\frac{dw}{dx} + z\left[3 - \frac{4}{3}\frac{z^2}{h^2} - \frac{16}{5}\frac{z^4}{h^4} - \frac{64}{7}\frac{z^6}{h^6}\right] * \phi(x) \quad (2)$$

$$w(x,\ z) = w(x)$$

'*u*' – Axial- displacement

'*w*'- Transverse displacement

2.2 Strains

Normal Strain:

$$\varepsilon_x = \frac{du}{dx} = -z\frac{d^2w}{dx^2} + z\left(3 - \frac{4}{3}\frac{z^2}{h^2} - \frac{16}{5}\frac{z^4}{h^4} - \frac{64}{7}\frac{z^6}{h6}\right)\frac{d\phi}{dx} \quad (3)$$

Shear Strain:

$$\gamma_{zx} = \frac{du}{dz} + \frac{dw}{dx} = \left(3 - 4\frac{z^2}{h^2} - 16\frac{z^4}{h^4} - 16\frac{z^6}{h^6}\right)\phi(x) \quad (4)$$

2.3 Stresses

Normal Stress:

$$\sigma_x = E\varepsilon_x = -Ez\frac{d^2w}{dx^2} + Ez\left(3 - \frac{4}{3}\frac{z^2}{h^2} - \frac{16}{5}\frac{z^4}{h^4} - \frac{64}{7}\frac{z^6}{h^6}\right)\frac{d\phi}{dx} \tag{5}$$

Shear Stress:

$$\tau_{zx} = G\gamma_{zx} = G\left(3 - 4\frac{z^2}{h^2} - 16\frac{z^4}{h^4} - 16\frac{z^6}{h^6}\right)\phi(x) \tag{6}$$

2.4 Governing-Equations Derivation and Associated Boundary Conditions

From Equations (2) to (6), Virtual work Principle's, the boundary conditions and governing differential equations for the beam will be

$$b\int_{x=0}^{x=L}\int_{z=-h/2}^{z=+h/2}\left(\sigma_x\delta\varepsilon_x + \tau_{zx}\,\delta\gamma_{zx}\right)dx\,dz = \int_{x=0}^{x=L} q(x)\,\delta w\,dx \tag{7}$$

Integrating equation (7), successively, a couple of equilibrium equations can be obtained.

$$EI\frac{d^4w}{dx^4} - 2.67EI\frac{d^3\phi}{dx^3} - q(x) = 0 \tag{8}$$

$$2.67EI\frac{d^3w}{dx^3} - 7.2EI\frac{d^2\phi}{dx^2} + 6GA\phi = 0 \tag{9}$$

Where, $A_0 = 2.67$, $B_0 = 7.2$, $C_0 = 6$

3.0 Illustrative Example

The following cantilever beam (Figure 1) with udl is considered for the affirmation purpose, the properties of beam material are 7800 kg/m^3, E = 210,000 MPa, μ = 0.3

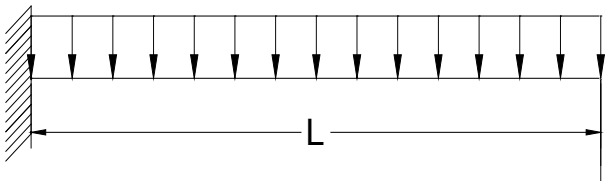


Figure 1. Cantilever deep beam carrying uniformly distributed load q0 (x).

ρ - density, E - modulus of elasticity, μ - Poisson's ratio

$\rho = q_9(x)$

The expressions for $w(x)$ and $\phi(x)$ for cantilever boundary condition is as follows.

$$w(x) = \frac{q_0L^4}{120EI}\left\{\left(\frac{x^5}{L^5} - 10\frac{x^3}{L^3} + 20\frac{x^2}{L^2}\right) + \frac{B_0}{C_0}\frac{E}{G}\frac{h^2}{L^2}\left(5\frac{x^2}{L^2} - \frac{5}{3}\frac{x^3}{L^3}\right) + 5\frac{A_0^2}{C_0}\frac{E}{G}\frac{h^2}{L^2}\left[\frac{\cosh\lambda x - \sinh\lambda x}{\lambda L} + \frac{x}{L} - \frac{1}{\lambda L}\right]\right\} \tag{10}$$

$$\phi(x) = -\frac{1}{2}\frac{A_0}{C_0}\frac{q_0L}{GA}\left(\cosh\lambda x - \sinh\lambda x + \frac{x^2}{L^2} - 1\right) \tag{11}$$

Axial displacement, u

$$u = \frac{q_0h}{Eb}\left\{\begin{array}{l} -\frac{z}{h}\frac{L^3}{h^3}\left[\left(\frac{1}{2}\frac{x^4}{L^4} - 3\frac{x^2}{L^2} + 4\frac{x}{L}\right) + \frac{B_0}{C_0}\frac{E}{G}\frac{h^2}{L^2}\left(\frac{x}{L} - \frac{1}{2}\frac{x^2}{L^2}\right) + \frac{1}{2}\frac{A_0^2}{C_0}\frac{E}{G}\frac{h^2}{L^2}\left(\sinh\lambda x - \cosh\lambda x + 1\right)\right] \\ -\frac{1}{2}\frac{A_0}{C_0}\frac{E}{G}\frac{L}{h}\frac{z}{h}\left(3 - \frac{4}{3}\frac{z^2}{h^2} - \frac{16}{5}\frac{z^4}{h^4} - \frac{64}{7}\frac{z^6}{h^6}\right)\left(\cosh\lambda x - \sinh\lambda x + \frac{x^2}{L^2} - 1\right) \end{array}\right\} \tag{12}$$

Axial stress σ_x,

$$\sigma_x = \frac{q_0}{b}\left\{\begin{array}{l} -\frac{z}{h}\frac{L^2}{h^2}\left[\left(2\frac{x^3}{L^3} - 6\frac{x}{L} + 4\right) + \frac{B_0}{C_0}\frac{E}{G}\frac{h^2}{L^2}\left(1 - \frac{x}{L}\right) + \frac{1}{2}\frac{A_0^2}{C_0}\frac{E}{G}\frac{h^2}{L^2}\lambda L\left(\cosh\lambda x - \sinh\lambda x\right)\right] \\ -\frac{1}{2}\frac{A_0}{C_0}\frac{E}{G}\frac{z}{h}\left(3 - \frac{4}{3}\frac{z^2}{h^2} - \frac{16}{5}\frac{z^4}{h^4} - \frac{64}{7}\frac{z^6}{h^6}\right)\left(\lambda L\sinh\lambda x - \lambda L\cosh\lambda x + 2\frac{x}{L}\right) \end{array}\right\} \tag{13}$$

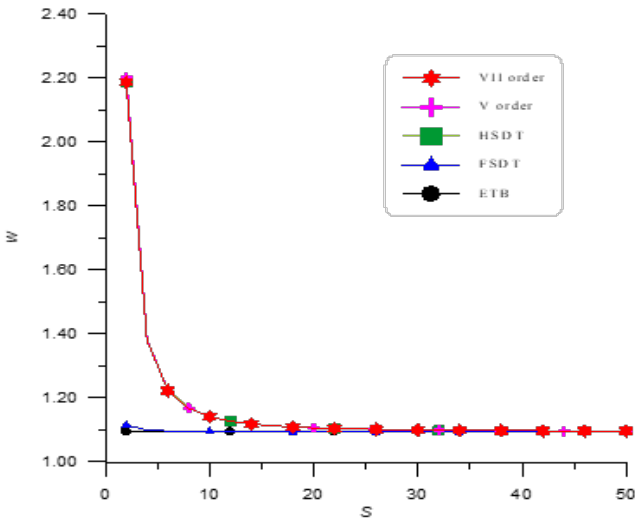


Figure 2. Changes in Transverse displacement ($\bar{w}$) through beam thickness.

Transverse shear stress τ_{zx}^{CR} obtained from the consecutive relationship

$$\tau_{zx}^{CR} = -\frac{1}{2}\frac{q_0}{b}\frac{A_0}{C_0}\frac{L}{h}\left(3-4\frac{z^2}{h^2}-16\frac{z^4}{h^4}-64\frac{z^6}{h^6}\right)\left(\cosh\lambda x-\sinh\lambda x+\frac{x^2}{L^2}-1\right) \quad (14)$$

Transverse shear stress τ_{zx}^{EE} from equilibrium equation

$$\tau_{zx}^{EE} = \frac{q_0}{b}\left\{\begin{array}{l}-\frac{1}{8}\frac{L}{h}\left(1-4\frac{z^2}{h^2}\right)\left[\left(6\frac{x^2}{L^2}-6\right)-\frac{B_0}{C_0}\frac{E}{G}\frac{h^2}{L^2}+\frac{1}{2}\frac{A_0^2}{C_0}\frac{E}{G}\frac{h^2}{L^2}\lambda^2L^2\left(\sinh\lambda x-\cosh\lambda x\right)\right]\\+\frac{1}{2}\frac{A_0}{C_0}\frac{E}{G}\frac{h}{L}\left(\frac{3}{2}\frac{z^2}{h^2}-\frac{1}{3}\frac{z^4}{h^4}-\frac{8}{15}\frac{z^6}{h^6}-\frac{8}{7}\frac{z^8}{h^8}-\frac{1147}{3360}\right)\left(\lambda^2L^2\cosh\lambda x-\lambda^2L^2\sinh\lambda x+2\right)\end{array}\right\} \quad (15)$$

4.0 Results

The outcomes for displacement along with stresses are furnished in the non-dimensional pattern as follows;

$$\bar{u}=\frac{E.b.u}{q_0h},\quad \bar{w}=\frac{10Ebh^3w}{q_0L^4},\quad \bar{\sigma}_x=\frac{b\sigma_x}{q_0},\quad \bar{\tau}_{zx}=\frac{b\tau_{zx}}{q_0}$$

Table 1. Non-dimensional parameters, Axial displacement '$\bar{u}$', transverse deflection '$\bar{w}$', axial Stress '$\bar{\sigma}_x$' transverse shear stresses '$\bar{\tau}_{zx}^{CR}$' and $\bar{\tau}_{zx}^{EE}$ at x = 0.25L having aspect ratio 'S' 2

Source	Model	$\bar{w}$	$\bar{u}$	$\bar{\sigma}_x$	$\bar{\tau}_{zx}^{CR}$	$\bar{\tau}_{zx}^{EE}$
Present	VII order	2.1870	4.2245	6.5687	1.2515	1.4036
Ghugal Y M[34]	V order	2.2034	4.2480	6.5657	1.3047	1.4016
Krishna Murty[17]	HSDT	2.1875	4.2895	6.5720	1.4060	1.3997
Timoshenko[8]	FSDT	1.1150	3.2578	5.0625	0.0652	1.4062
Bernoulli-Euler[1,3,4]	ETB	1.0947	3.2578	5.0625	-	1.4062

Table 2. Non-dimensional parameters Axial displacement '$\bar{u}$', transverse deflection '$\bar{w}$', axial Stress '$\bar{\sigma}_x$' transverse shear stresses '$\bar{\tau}_{zx}^{CR}$' and $\bar{\tau}_{zx}^{EE}$ at x = 0.25L having aspect ratio 'S' 4

Source	Model	$\bar{w}$	$\bar{u}$	$\bar{\sigma}_x$	$\bar{\tau}_{zx}^{CR}$	$\bar{\tau}_{zx}^{EE}$
Present	VII order	2.1870	27.9960	21.7561	2.5031	2.8113
Ghugal Y M[26]	V order	2.2034	28.0430	21.7532	2.6096	2.8102
Krishna Murty[10]	HSDT	2.1875	28.1259	21.7590	2.8125	2.8096
Timoshenko[8]	FSDT	1.1150	26.0625	20.2500	0.1402	2.8125
Bernoulli-Euler[1,3,4]	ETB	1.0947	26.0625	20.2500	-	2.8125

Table 3. Non-dimensional parameters Axial displacement '$\bar{u}$', transverse deflection '$\bar{w}$', axial Stress '$\bar{\sigma}_x$' transverse shear stresses '$\bar{\tau}_{zx}^{CR}$' and $\bar{\tau}_{zx}^{EE}$ at x = 0.25L having aspect ratio 'S' 6

Source	Model	$\bar{w}$	$\bar{u}$	$\bar{\sigma}_x$	$\bar{\tau}_{zx}^{CR}$	$\bar{\tau}_{zx}^{EE}$
Present	VII order	1.2232	90.861	47.068	3.7546	4.2179
Ghugal Y M[34]	V order	1.2242	90.931	47.065	3.9142	4.2172
Krishna Murty[17]	HSDT	1.2242	91.056	47.071	4.2187	4.2168
Timoshenko[8]	FSDT	1.0970	87.960	45.562	0.2152	4.2187
Bernoulli-Euler[1,3,4]	ETB	1.0947	87.960	45.562	-	4.2187

Table 4. Non-dimensional parameters Axial displacement '$\bar{u}$', transverse deflection '$\bar{w}$', axial Stress '$\bar{\sigma}_x$' transverse shear stresses '$\bar{\tau}_{zx}^{CR}$' and $\bar{\tau}_{zx}^{EE}$ at x = 0.25L having aspect ratio 'S' 8

Source	Model	$\bar{w}$	$\bar{u}$	$\bar{\sigma}_x$	$\bar{\tau}_{zx}^{CR}$	$\bar{\tau}_{zx}^{EE}$
Present	VII order	1.1675	212.367	82.506	5.0062	5.6243
Ghugal Y M[34]	V order	1.1680	212.460	82.503	5.2190	5.6238
Krishna Murty[17]	HSDT	1.1681	212.626	82.508	5.6250	5.6235
Timoshenko[8]	FSDT	1.0960	208.500	81.000	0.2902	5.6250
Bernoulli-Euler[1,3,4]	ETB	1.0947	208.500	81.000	-	5.6250

Table 5. Non-dimensional parameters Axial displacement '$\bar{u}$', transverse deflection '$\bar{w}$', axial Stress '$\bar{\sigma}_x$' transverse shear stresses '$\bar{\tau}_{zx}^{CR}$' and $\bar{\tau}_{zx}^{EE}$ at x = 0.25L having aspect ratio 'S' 10

Source	Model	$\bar{w}$	$\bar{u}$	$\bar{\sigma}_x$	$\bar{\tau}_{zx}^{CR}$	$\bar{\tau}_{zx}^{EE}$
Present	VII order	1.1415	412.060	128.068	6.2578	7.0308
Ghugal Y M[26]	V order	1.1418	412.177	128.065	6.5238	7.0303
Krishna Murty[10]	HSDT	1.1419	412.385	128.071	7.0313	7.0301
Timoshenko[8]	FSDT	1.0955	407.226	126.562	0.3652	7.0313
Bernoulli-Euler[1,3,4]	ETB	1.0947	407.226	126.526	-	7.0313

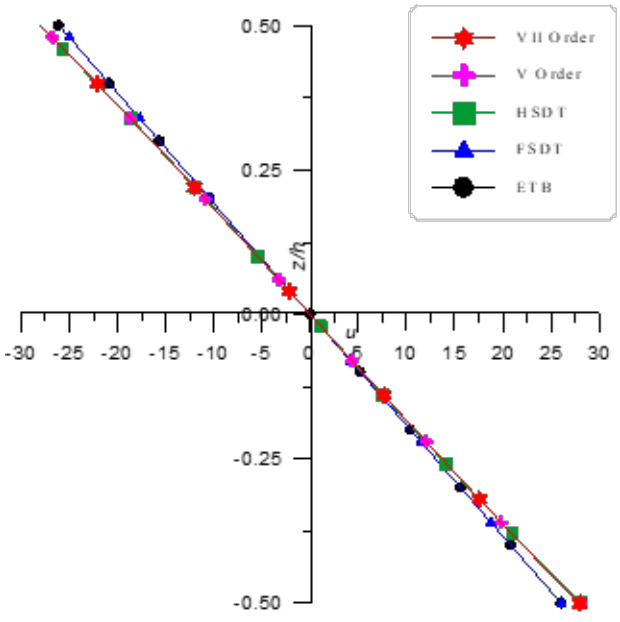


Figure 3. Aspect ratio 4: Changes in Axial displacement ($\bar{u}$) through beam thickness.

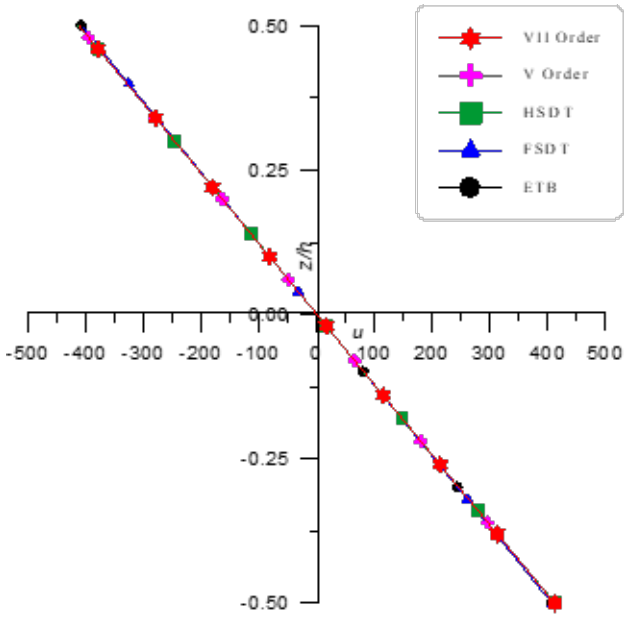


Figure 4. Aspect ratio 10: Changes in Axial displacement ($\bar{u}$) through beam thickness.

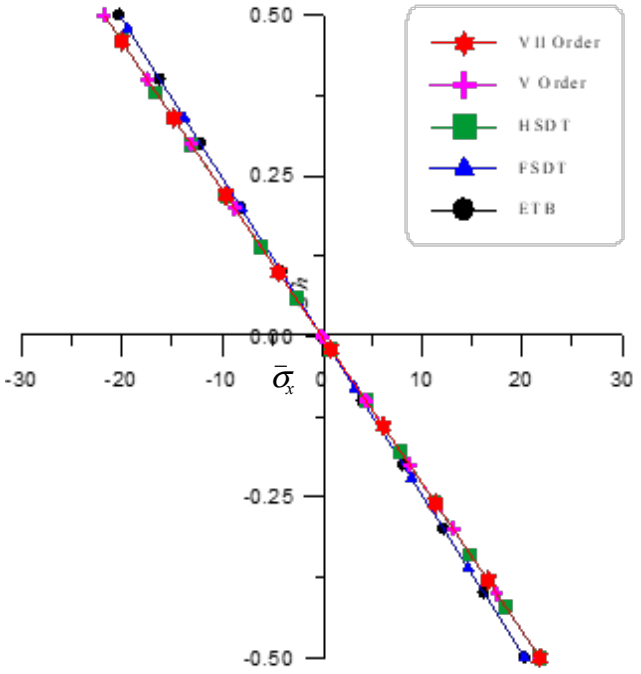


Figure 5. Aspect ratio 4: Changes in Axial Stress ($\bar{\sigma}_x$) through beam thickness.

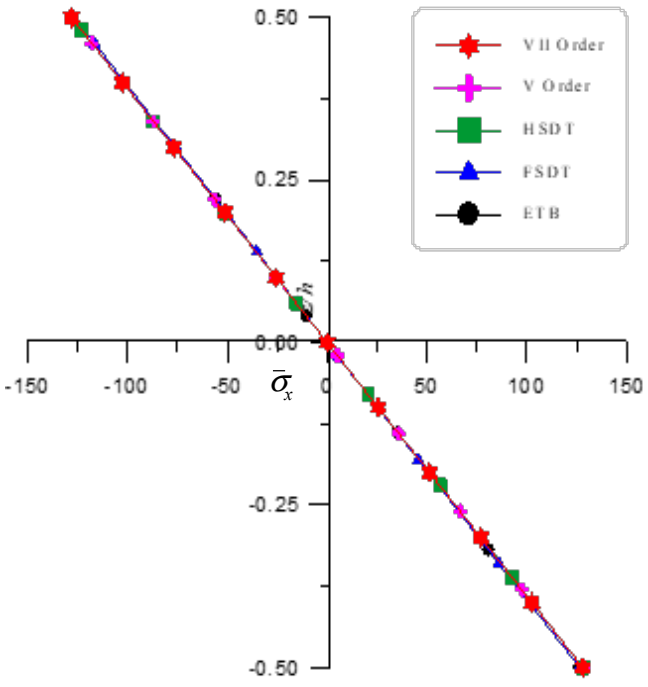


Figure 6. Aspect ratio 10: Changes in Axial Stress ($\bar{\sigma}_x$) through beam thickness.

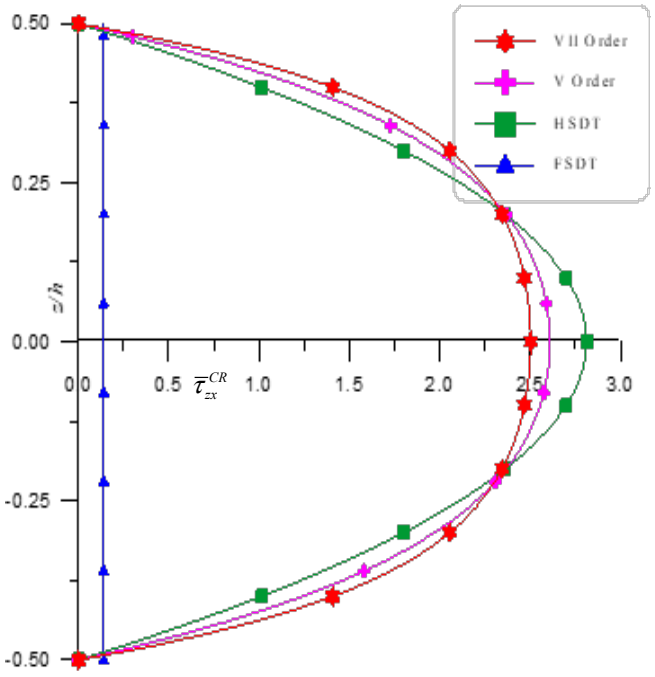


Figure 7. Aspect ratio 4: Changes in Transverse shear stresses ($\bar{\tau}_{zx}^{CR}$) (Constitutive relationship) through beam thickness.

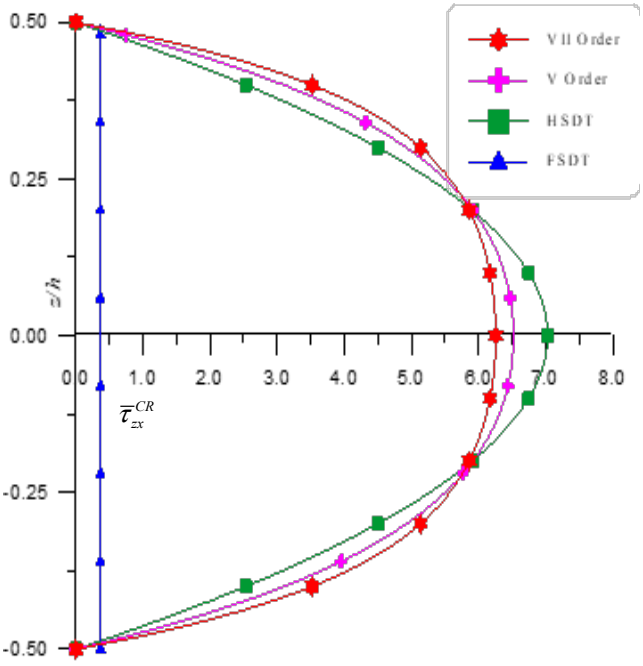


Figure 8. Aspect ratio 10: Changes in Transverse shear stresses ($\bar{\tau}_{zx}^{CR}$) (Constitutive relationship) through beam thickness.

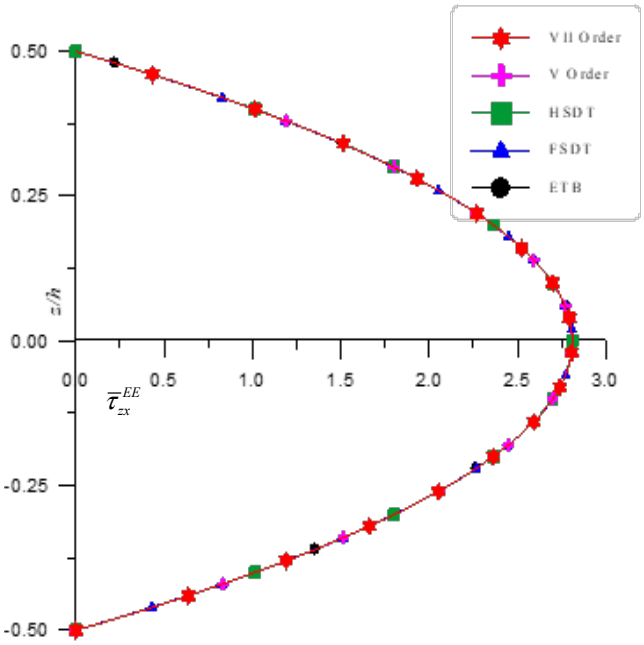


Figure 9. Aspect ratio 4: Changes in Transverse shear stresses ($\overline{\tau}_{zx}^{EE}$) (Equilibrium equation) through beam thickness.

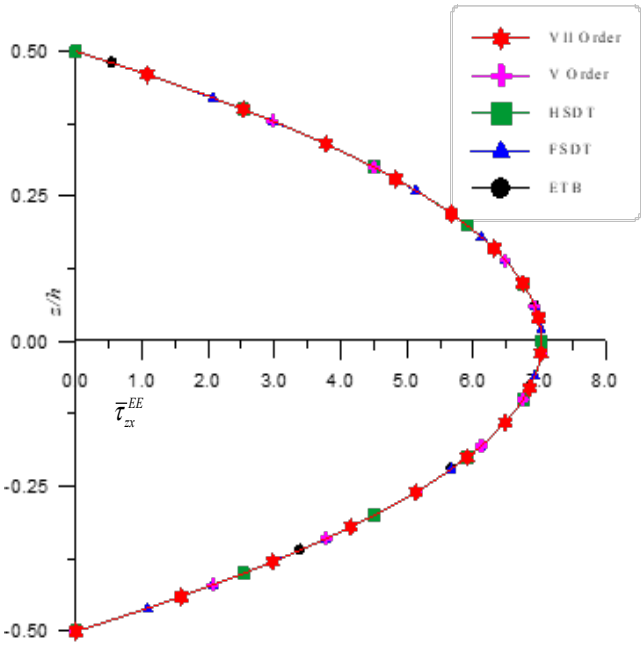


Figure 10. Aspect ratio 10. Changes in Transverse shear stresses ($\overline{\tau}_{zx}^{EE}$) (Equilibrium equation) through beam thickness.

5.0 Conclusion

Table 5 outlines the comparison of axial displacements, Transverse displacements, normal stresses and shear stresses of a cantilever deep beam carrying uniformly distributed load with load intensity q_0 (x).

The following conclusions are drawn

- The axial displacement obtained by this theory shows agreement with all other theories.
- The transverse displacements match with the other theories except ETB. The deflections given by ETB are lower than that of the present theory due to the ignorance of shear deformation effects. The graphs for the transverse displacement (Figure 2) depict that the values of $\overline{w}$ Reduces as the aspect ratio increases.
- Axial stresses are similar to all the considered theories, however, the Bernoulli-Euler theory and Timoshenko theory yield lower values.
- The Transverse shear stresses meet the no shear-stress boundary condition on top as well as the beam's bottom face of the beam. Stress distribution over the depth of the beam is in line with the other refined SD theories.
- The transverse displacement '$\overline{w}$' remains constant for all the aspect ratios for ETB, while for FSDT, HSDT VI order and VII order, it goes on reducing with an increase in aspect ratio. As the aspect ratios surpass 20 all the refined theories tend to match the values of classical beam theory (ETB). It is also observed that with a slight increase in the length-to-width ratio, there is a considerable increase in the non-dimensional values of Axial displacement '$\overline{u}$', transverse deflection '$\overline{w}$', axial Stress '$\overline{\sigma}_x$' transverse shear stresses '$\overline{\tau}_{zx}^{CR}$' and $\overline{\tau}_{zx}^{EE}$ As depicted in the graphs and result tables.
- In current theory, the requirement of shear correction coefficient is set aside.
- In practical applications, this theory can be instrumental in designing strong and reliable structures, particularly in cases where shear deformation effects cannot be ignored, such as in deep beams, bridges, and heavy-duty construction components.

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