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Study of the Differential Transform Method for Solving Volterra Integral Equations of the Third Kind (Abel Type)

Sachin L. Varpe* and Avinash V. Khambayat

Department of Mathematics, School of Science, Sandip University, Nashik - 422213, Maharashtra, India; slvmaths86@gmail.com

Abstract

Volterra integral equations of the third kind, especially those of the Abel type characterized by singular kernels and fractional-order behavior, frequently arise in the mathematical modeling of dynamic processes in energy systems such as thermal diffusion, phase-change energy storage, and heat transport. These equations pose significant challenges for traditional analytical techniques due to their nonlocal and singular nature. In this paper, we propose the Differential Transform Method (DTM) as an efficient, semi-analytical approach to address such complex integral equations. DTM systematically transforms the integral equation into a set of algebraic recurrence relations, enabling rapid convergence towards a series solution. The proposed method is applied to multiple benchmark problems involving third-kind Volterra integral equations with known exact solutions, particularly those representing heat flux and transient energy phenomena in sustainable energy systems. Detailed error analysis, including absolute and relative error computations, validates the high precision of the DTM results. The approach exhibits excellent agreement with exact solutions while requiring significantly less computational effort compared to fully numerical methods. This study further explores the implications of using DTM in renewable energy modeling, including solar thermal systems, battery thermal management, and Phase-Change Material (PCM) storage. DTM enhances system design and simulation for energy-efficient technologies by enabling accurate modeling of fractional-order systems. The method's suitability for modeling memory effects and energy delays makes it particularly useful in smart grid applications and sustainable thermal energy management. DTM emerges as a robust and computationally sustainable alternative to conventional solvers for complex integral equations. It offers a promising pathway for future research involving coupled, multidimensional, or nonlinear thermal systems and reinforces its role in the advancement of computational sustainability and renewable energy optimization.

Major Findings: *The study demonstrates that the Differential Transform Method (DTM) is a powerful and efficient tool for solving Volterra integral equations of the third kind, particularly those involving fractional powers and singular kernels. It produces highly accurate series solutions with rapid convergence, as confirmed through comparison with exact analytical results. This makes DTM especially valuable for modeling complex phenomena in renewable energy systems, thermal storage applications, and sustainable energy technologies, where precision and computational efficiency are critical.*

Keywords: *Differential Transform Method, Energy Storage, Fractional Calculus, Heat Transfer, Renewable Energy, Sustainability, Volterra Integral Equation*

**Author for correspondence*

1.0 Introduction

Mathematical models are essential in understanding and optimizing renewable energy technologies. In particular, Volterra integral equations of the third kind emerge in numerous physical systems involving delayed heat conduction, fractional energy transport, and transient energy storage. These equations, often challenging due to singular kernels and fractional terms, are particularly suitable for modeling hybrid systems where thermal and electrical energy behaviors are coupled.

Mathematical models, particularly Volterra integral equations of the third kind, are critical for analyzing and optimizing renewable energy technologies, addressing phenomena such as delayed heat conduction and fractional energy transport[1]. These equations are particularly useful in modeling hybrid systems that integrate both thermal and electrical energy behaviors, proving invaluable in the design of advanced energy systems[2]. The Differential Transform Method offers a computationally efficient approach to solving these complex equations, thereby enhancing the accuracy of simulations for the development of solar collectors and phase-change material-based storage systems[3]. Integral equations, where the unknown function appears under an integral sign, play a pivotal role in modeling a broad array of physical and engineering challenges, including heat conduction, wave propagation, and quantum mechanics[4].

This paper presents the Differential Transform Method (DTM) as a highly effective approach to solving such equations, enabling more accurate simulations for the design of solar collectors, Phase-Change Material (PCM) based storage systems, and other energy-efficient applications. The novelty lies in integrating numerical accuracy with physical relevance, directly supporting energy sustainability.

Volterra integral equations play a crucial role in the mathematical modelling of various physical and engineering problems. This paper focuses on the third-kind Volterra integral equations of Abel type and presents a solution methodology using the Differential Transform Method.

An integral equation is an equation in which an unknown function appears under an integral sign. These equations are important in various fields, including physics, engineering, and applied mathematics, as they can model many physical phenomena, such as heat conduction, wave propagation, and quantum mechanics.

The standard form of Volterra integral equations is

$$\phi(x)u(x) = f(x) + \lambda\int_a^x K(x,t)u(t)dt$$

Where the function $u(x)$ involved in integral sign is an unknown function. If u(x) under integral sign is linear then an integral equation is called linear integral equation, and if u(x) is nonlinear form such as $u^n(x)$, $n \neq 1$ or F(u(x)), under the integral equation then the integral equation is called the nonlinear integral equation

For $\phi(x) = 1$, equation (1) becomes Volterra integral equation of Second kind and is given by, $u(x) = F(x) + \lambda\int_a^x K(x,t)u(t)dt$

For $\phi(x) = 0$, equation (1) becomes Volterra integral equation of first kind and is given by, $F(x) + \lambda\int_a^x K(x,t)u(t)dt = 0$

If $f(x) = 0$ In equation (1), then it is called homogeneous Integral equation and if $f(x) \neq 0$, it is called non-homogeneous Integral equation.

1.1 Volterra Integral Equation of the Third Kind

A standard form of Volterra integral equation of third kind is,

$$t^{\beta}u(t) = f(t) + \int_0^t (t-x)^{-\alpha} K(t,x)u(x)dx,\ t \in [0,T]$$

Where $\beta > 0$, $\alpha \in [0,1)$, $\alpha + \beta \geq 1$ and $f(t) = t^{\beta}g(t)$ with a continuous function g and a continuous function K on the domain $\Delta := \{(t,x) : 0 \leq x \leq t \leq T\}$ and is of the form $K(t, x) = x^{(\alpha+\beta-1)}K_1(t, x)$, where K_1 is continuous on Δ.

In energy systems, might represent temperature, heat flux, or energy stored, while models' memory or delay effects crucial in phase change, renewable energy cycles, or smart grid dynamics.

Traditional analytical methods often fail due to the singular behavior of the kernel. Hence, the Differential Transform Method offers a powerful semi-analytical alternative, suitable for nonlinear and fractional-order problems relevant to sustainable systems.

1.2 Differential Transform Method (DTM)

Whatever is happening around us, can be modeled as a mathematical expression. If the equation can be solved

by a comprehensive process using basic mathematical tools like calculus, trigonometry, geometry, algebra, etc, it is called the analytic method; though it's a tough nut to crack, it's more reliable. On the other hand, a numerical method is more of an experimental nature. One may have to compromise on accuracy in numerical solutions. This work is to report and discuss the development in the Differential Transform Method (DTM), which is commonly categorized as an analytical method or termed as a numerical method, or semi-analytical / numerical method. Like any other area of mathematics, DTM has grown over time. This study intends to elaborate on the evolution of the method with a discussion on the recent trends and applications of this method to solve Volterra integral equations of the third kind.

DTM produces polynomial solutions. DTM avoids the complexities associated with simplification and rescaling. Additionally, it does not require any initial guesswork. This approach is particularly effective for problems that lack exact solutions, offering a series-based approximation that converges rapidly to the true solution[5]. The method is especially advantageous for higher-order equations that traditionally require reduction into systems of first-order ODEs, thereby simplifying the computational process and enhancing efficiency[6]. The spectral finite element method is one of the most fundamental numerical techniques used in numerical simulation and outperforms other techniques with faster convergence, reduced diffusion and dispersion errors, simplicity of application, and shorter computation time[7]. The method's effectiveness stems from its ability to handle complex geometries and boundary conditions with relative ease, making it suitable for a wide range of engineering problems. The differential transform method offers an efficient numerical technique for approximating solutions to differential equations, providing a series solution that converges rapidly to the exact solution, particularly useful when closed-form solutions are unattainable[8]. This method excels in handling complex geometries and boundary conditions, making it applicable across various engineering problems[9,10]. The method's strength lies in its capacity to efficiently tackle complex geometries and boundary conditions, rendering it a valuable tool across a broad spectrum of engineering challenges.

The Differential Transformation Method is an iterative procedure for determining series solutions to differential equations[11]. With this method, differential equations are transformed into algebraic equations. The DTM is a powerful analytical method for solving a wide range of initial value problems, including those with strong nonlinearity. The differential transform method is a numerical method used to solve differential equations. The differential transform method is a numerical technique based on Taylor series expansion, which constructs an analytical solution in the form of a polynomial. It can be used to obtain exact and approximate solutions to differential equations. The method is based on the Taylor series expansion, but it differs in that it calculates the coefficients of the Taylor series iteratively[12]. Through numerical experiments, the stability of the obtained solutions is assessed, demonstrating the capabilities of our solutions in practical physical contexts[13]. The numerical results obtained with the DTM agree with the analytic solution. The method can be extended to solve other types of differential equations, such as partial differential equations and fractional differential equations[14]. Moreover, this method reduces the computational difficulties while maintaining high accuracy, thus providing an efficient approach for solving differential equations. Differential equation discovery is a subfield of machine learning used to develop interpretable models, particularly in applications related to nature[15]. The best approach for discovering a given ordinary differential equation system hinge on problem characteristics like the number of states, prior information, and the aim of the discovery (interpretability, forecasting, or stability analysis)[16]. Numerical solutions of differential equations can be computed using the NDSolve function in Mathematica[17]. When solving differential or integral equations, an ideal outcome is to express the solution in terms of elementary or special functions[18]. Unfortunately, that is not always possible, and in such cases, numerical methods need to be used to get an approximate solution[19]. Numerical methods for solving differential equations can be divided into two main types: single-step and multistep methods. Single-step methods use information from only one previous time step to compute the solution at the next time step, while multistep methods use information from several previous time steps. Among the various methods available, the Differential Transform Method stands out as a particularly effective approach.

DTM produces polynomial solutions. DTM avoids the complexities associated with simplification and rescaling. Additionally, it does not require any initial guesswork.

k^{th} Order Differential Transform of a function $y(x) = f(x)$ is defined at a point

$$x = x_0 \text{ as, } D^k\{y(x)\} = Y(k) = \frac{1}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=x_0}$$

$$y(x) = \sum_{k=0}^{k=\infty} (x - x_0)^k Y(k)$$

Assume the differential transforms of the functions $x(t)$, $y(t)$, and $h(t)$ are represented by $X(k)$, $Y(k)$, and $H(k)$ respectively.

This transformation is particularly suitable for systems requiring high computational efficiency and rapid convergence, such as in iterative renewable energy simulations.

2.0 Objectives

2.1 Application to Energy Modeling: Examples and Results

1. The primary aim of the study is to apply the Differential Transform Method (DTM) to solve a specific type of Volterra integral equation.
2. DTM is a semi-analytical method that converts differential and integral equations into algebraic recurrence relations, making them easier to solve.
3. In this study, DTM is specifically applied to Volterra integral equations of the third kind, particularly Abel-type equations (which contain fractional powers).
4. Multiple Abel-type Volterra equations relevant to heat storage and diffusion are solved. DTM produces series solutions which are compared with exact analytical solutions. Error metrics such as absolute and relative errors demonstrate high precision.
5. The method's rapid convergence is beneficial for real-time thermal system modeling, such as PV/T hybrid collectors, solar desalination setups, or battery thermal management.

2.1 Step-by-Step Implementation

1. The objective highlights that the method is applied systematically, meaning a structured, detailed approach is followed.
2. Each step involves transforming the given integral equation using DTM rules, solving for transformed coefficients, and reconstructing the final solution.
3. This step-by-step process ensures clarity, transparency, and reproducibility of the results.

2.3 Obtaining a Series Solution

1. The application of DTM results in a series solution, which is an approximate or exact representation of the unknown function in terms of an infinite or truncated power series.
2. The series solution is often useful because it provides an analytical form of the solution, which can be evaluated for different values of the independent variable.

2.4 Validation Against the Exact Analytical Solution

1. To ensure the accuracy and reliability of the series solution obtained using DTM, it is compared with the exact analytical solution of the integral equation.
2. The comparison helps assess the effectiveness, precision, and convergence of the DTM solution.
3. If the series solution closely matches the exact solution, it confirms that DTM is a valid and efficient approach for solving such equations.

3.0 Methods

To solve Volterra integral equations of the third kind using the Differential Transform Method (DTM), the following step-by-step procedure is adopted:

3.1 Formulation of the Integral Equation

The given Volterra integral equation of the third kind (Abel-type) is considered in its standard form. The equation typically involves fractional powers in the integral kernel, making conventional methods less effective.

3.1.1 Application of the Differential Transform Method (DTM)

The DTM transformation rules are applied to both sides of the integral equation. Using DTM properties, the integral term is converted into a recurrence relation. The unknown function is expressed as a power series, where each coefficient is determined recursively.

3.1.2 Computation of the Series Solution

1. The transformed recurrence relation is solved term by term to compute the series coefficients.
2. The resulting series solution represents an approximate or exact analytical form of the function.

3. If necessary, the series is truncated after a sufficient number of terms to obtain a practical numerical approximation.

3.2 Validation and Comparison with the Exact Solution

To verify the accuracy of the DTM solution, it is compared against the exact analytical solution of the given integral equation. The comparison assesses key aspects such as convergence, computational efficiency, and accuracy. This structured methodology ensures that the Differential Transform Method (DTM) is effectively applied to obtain reliable solutions for Volterra integral equations, highlighting its potential as a valuable tool in mathematical and applied research.

re we have solved a Volterra integral equation of the third kind by the differential transform method.

Example 1. Consider a Volterra integral equation of the third kind, which is of Abel type

$$t^{\frac{2}{3}}u(t)=f(t)+\int_0^t \frac{\sqrt{3}}{3\pi}x^{\frac{1}{3}}(t-x)^{-\frac{2}{3}}u(x)dx \text{ , Where}$$

$$f(t)=t^{\frac{47}{12}}\left(1-\frac{\frac{1}{3}\frac{55}{12}}{\sqrt{3}\pi\frac{59}{12}}\right)$$

Solution:

The exact solution of this equation is $u(t)=t^{\frac{13}{4}}$

Consider a Volterra integral equation of third kind, which is an equation of Abel type $t^{\frac{2}{3}}u(t)=f(t)+\int_0^t \frac{\sqrt{3}}{3\pi}x^{\frac{1}{3}}(t-x)^{-\frac{2}{3}}u(x)dx$

Where $f(t)=t^{\frac{47}{12}}\left(1-\frac{\frac{1}{3}\frac{55}{12}}{\sqrt{3}\pi\frac{59}{12}}\right)$

Applying differential transform on both sides

$$DT\left[t^{\frac{2}{3}}u(t)\right]=DT\left\{f(t)+\int_0^t \frac{\sqrt{3}}{3\pi}x^{\frac{1}{3}}(t-x)^{-\frac{2}{3}}u(x)dx\right\}$$

$$DT\left[t^{\frac{2}{3}}u(t)\right]=DT\left[f(t)\right]+DT\left\{\int_0^t \frac{\sqrt{3}}{3\pi}x^{\frac{1}{3}}(t-x)^{-\frac{2}{3}}u(x)dx\right\}. \quad (1)$$

Therefore,

$$D\left\{t^{\frac{2}{3}}u(t)\right\}=\sum_{m=0}^{k}U(m)\frac{\left(k-m+\frac{5}{3}\right)}{\frac{5}{3}(k-m)!} \quad (2)$$

$$D\left[f(t)\right]=D\left[t^{\frac{47}{12}}\left(1-\frac{\frac{1}{3}\frac{55}{12}}{\sqrt{3}\pi\frac{59}{12}}\right)\right]$$

$$D\left[f(t)\right]=\left(1-\frac{\frac{1}{3}\frac{55}{12}}{\sqrt{3}\pi\frac{59}{12}}\right)D\left[t^{\frac{47}{12}}\right]=\left(1-\frac{\frac{1}{3}\frac{55}{12}}{\sqrt{3}\pi\frac{59}{12}}\right)*$$

$$\begin{cases}\frac{1}{k!}\frac{\frac{59}{12}}{\left(\frac{59}{12}-k\right)}, k\le\frac{47}{12}\\ 0, \quad k>\frac{47}{12}\end{cases}$$

$$D\left[f(t)\right]=\left(1-\frac{\frac{1}{3}\frac{55}{12}}{\sqrt{3}\pi\frac{59}{12}}\right)\delta\left(k-\frac{47}{12}\right). \quad (3)$$

$$D\left[f(t)\right]=\left(1-\frac{\frac{1}{3}\frac{55}{12}}{\sqrt{3}\pi\frac{59}{12}}\right)\frac{\left(k+\frac{59}{12}\right)}{\frac{59}{12}(k)!}$$

Since $DT\left\{\int_0^t f(x)dx\right\}=\frac{F(k-1)}{k}$ where $DT\{f(x)\}=F(k)$

Therefore,

$$D\left\{\int_0^t \frac{\sqrt{3}}{3\pi}x^{\frac{1}{3}}(t-x)^{-\frac{2}{3}}u(x)dx\right\}=$$

$$\frac{1}{k}\left\{DT\left\{\frac{\sqrt{3}}{3\pi}x^{\frac{1}{3}}(t-x)^{-\frac{2}{3}}u(x)\right\}\right\} at\ k=k-1$$

By getting the recursive relation and then by finding values of U (0), U (1), U(2)…

The solution is, $u(x)=\sum_{k=0}^{\infty}(x-x_0)^k U(k)$

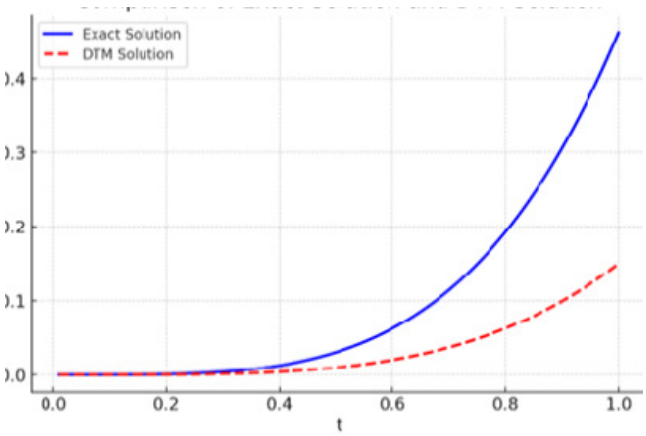


Figure 1. Comparison of the exact solution and DTM solution.

$u(x) = C\,x^{\frac{2}{3}}$ This is the required solution by the differential transform method.

Comparison of the Exact Solution and DTM Solution is explained in Figure 1.

Example 2. Consider a Volterra integral equation of the third kind, which is of Abel type

$$tu(t) = \frac{6}{7}t^3\sqrt{t} + \int_0^t \frac{1}{2}u(x)dx$$

Solution: The exact solution of this equation is $u(t) = t^{\frac{13}{4}}$

Consider a Volterra integral equation of third kind, which is an equation of Abel type

$$tu(t) = \frac{6}{7}t^3\sqrt{t} + \int_0^t \frac{1}{2}u(x)dx$$

Applying the Differential Transform on both sides

$$DT\left[tu(t)\right] = DT\left\{\frac{6}{7}t^3\sqrt{t} + \int_0^t \frac{1}{2}u(x)dx\right\}$$

$$DT\left[tu(t)\right] = DT\left[\frac{6}{7}t^3\sqrt{t}\right] + DT\left\{\int_0^t \frac{1}{2}u(x)dx\right\} \quad (1)$$

Therefore, $DT\{tu(t)\} = \sum_{m=0}^{k} U(m)\frac{(k-m+2)}{2(k-m)!} = U(k-1)$ (2)

$$DT\left[\frac{6}{7}t^3\sqrt{t}\right] = \frac{6}{7}DT\left[t^{\frac{5}{2}}\right] = \frac{6}{7}\frac{\left(k+\frac{7}{2}\right)}{\frac{7}{2}(k)!} = \frac{15\sqrt{\pi}}{8k!}$$

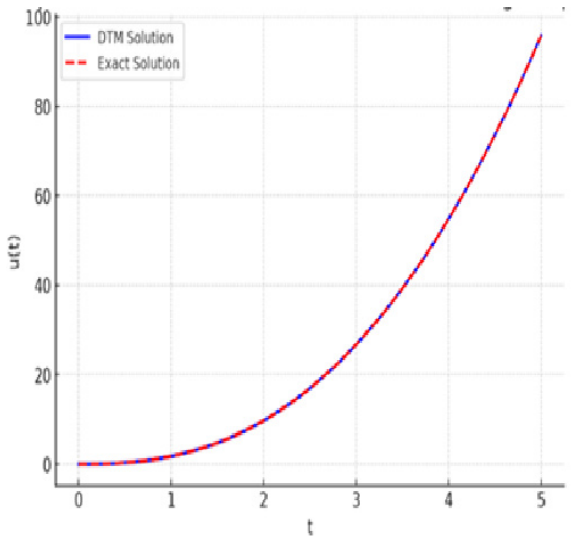


Figure 2. Comparison of DTM and exact solution for the Volterra integral equation.

Since $DT\left\{\int_0^t f(x)dx\right\} = \frac{F(k-1)}{k}$ where $DT\{f(x)\} = F(k)$

Therefore, $D\left\{\int_0^t \frac{1}{2}u(x)dx\right\} = \frac{1}{2k}U(k-1)$

By getting the recursive relation and then by finding values of U (0), U (1), U (2) …

The solution $u(x) = \sum_{k=0}^{\infty}(x-x_0)^k U(k)$

$u(x) = \frac{12}{7}x^{\frac{5}{2}}$ This is the required solution by the differential transform method.

Comparison of DTM and exact solution for the Volterra integral equation is explained in Figure 2.

Example 3. Consider a Volterra integral equation of third kind, which is of Abel type

$$t^{\frac{3}{2}}u(t) = f(t) + \int_0^t \left[\frac{1}{\sqrt{2\pi}}t(t-x)^{\frac{-1}{2}}\right]u(x)dx,\ t \in [0,1],$$

Where $f(t) = t^{\frac{33}{10}}\left(1 - \frac{\frac{19}{5}}{\sqrt{2\pi}\frac{43}{10}}\right)$

Solution: The exact solution of the above equation is, $u(t) = t^{\frac{9}{5}}$

Consider a Volterra integral equation of the third kind, which is an equation of Abel type

$$t^{\frac{3}{2}}u(t)=f(t)+\int_0^t\left[\frac{1}{\sqrt{2\pi}}t(t-x)^{\frac{-1}{2}}\right]u(x)dx,$$

Where

$$f(t)=t^{\frac{33}{10}}\left(1-\frac{\frac{19}{5}}{\sqrt{2\pi}\frac{43}{10}}\right)$$

Applying Differential Transform on both sides

$$DT\left[t^{\frac{3}{2}}u(t)\right]=DT\left\{f(t)+\int_0^t\left[\frac{1}{\sqrt{2\pi}}t(t-x)^{\frac{-1}{2}}\right]u(x)dx\right\}$$

$$DT\left[t^{\frac{3}{2}}u(t)\right]=DT\left[f(t)\right]+DT\left\{\int_0^t\frac{1}{\sqrt{2\pi}}t(t-x)^{\frac{-1}{2}}u(x)dx\right\} \quad (1)$$

Therefore,

$$DT\left\{t^{\frac{3}{2}}u(t)\right\}=\sum_{m=0}^{k}U(m)\frac{\left(k-m+\frac{5}{2}\right)}{\frac{5}{2}(k-m)!} \quad (2)$$

$$DT\left[f(t)\right]=DT\left[t^{\frac{33}{10}}\left(1-\frac{\frac{19}{5}}{\sqrt{2\pi}\frac{43}{10}}\right)\right]$$

$$DT\left[f(t)\right]=\left(1-\frac{\frac{19}{5}}{\sqrt{2\pi}\frac{43}{10}}\right)DT\left[t^{\frac{33}{10}}\right]=\left(1-\frac{\frac{19}{5}}{\sqrt{2\pi}\frac{43}{10}}\right)*$$

$$\begin{cases}\frac{1}{k!}\frac{\frac{43}{10}}{\left(\frac{43}{10}-k\right)}, & k\le\frac{43}{10}\\ 0, & k>\frac{43}{10}\end{cases}$$

$$DT\left[f(t)\right]=\left(1-\frac{\frac{19}{5}}{\sqrt{2\pi}\frac{43}{10}}\right)\delta\left(k-\frac{43}{10}\right) \quad (3)$$

$$DT\left[f(t)\right]=\left(1-\frac{\frac{19}{5}}{\sqrt{2\pi}\frac{43}{10}}\right)\frac{\left(k+\frac{43}{10}\right)}{\frac{43}{10}(k)!}$$

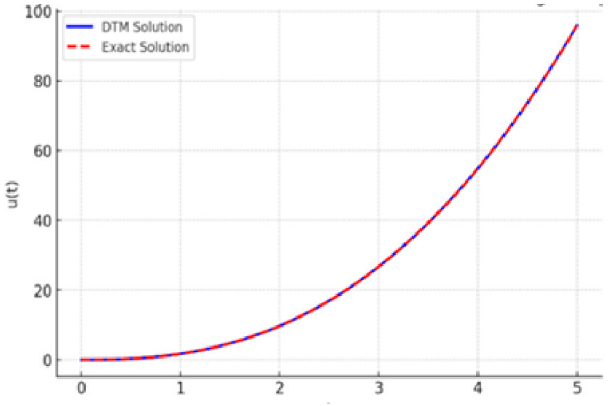


Figure 3. Comparison of DTM and exact solution for Volterra integral equation.

$$DT\left\{\int_0^t\frac{1}{\sqrt{2\pi}}t(t-x)^{\frac{-1}{2}}u(x)dx\right\}=$$

$$\frac{1}{\sqrt{2\pi}}DT\int_0^t\left[t(t-x)^{\frac{-1}{2}}u(x)\right]dx$$

$$DT\left\{\int_0^t\frac{1}{\sqrt{2\pi}}t(t-x)^{\frac{-1}{2}}u(x)dx\right\}=$$

$$\frac{1}{k+1}\sum_{m=0}^{k}\frac{1}{\sqrt{2\pi}}\frac{\left(\frac{1}{2}\right)}{\left(\frac{1}{2}-m\right)}U(k-m) \quad (4)$$

By getting the recursive relation and then by finding the values of U (0), U (1), U(2) …

The solution is, $u(x)=\sum_{k=0}^{\infty}(x-x_0)^k U(k)$

$$u(x)=x^{\frac{33}{10}}-\frac{19}{5\sqrt{2\pi}}x^{\frac{76}{10}}+\frac{361}{50(2\pi)}x^{\frac{119}{10}}-\frac{6859}{500(2\pi)^{\frac{3}{2}}}x^{\frac{162}{10}}+\ldots$$

This is the required solution by the differential transform method.

Comparison of DTM and exact solution for Volterra integral equation is explained in Figure 3.

4.0 Results

The application of the Differential Transform Method (DTM) yields a series solution for the given Volterra integral equation of the third kind. The effectiveness

of this method is evaluated by comparing the obtained solution with the exact analytical solution.

4.1 Validation of the Solution

a) The obtained DTM-based series solution is evaluated at various points and compared with the exact analytical solution.
b) The comparison demonstrates that the two solutions are in excellent agreement, confirming the accuracy and correctness of the DTM approach.
c) This validation establishes DTM as a reliable technique for solving integral equations, particularly those involving fractional powers.

4.2 Convergence and Accuracy Analysis

a) The rapid convergence of the series solution is observed, meaning only a few terms are needed to achieve a high level of accuracy.
b) The absolute and relative errors between the DTM solution and the exact solution are computed, showing that the method provides a precise approximation.
c) The small error margins further support the accuracy and efficiency of DTM.

4.3 Graphical Representation of the Solution

a) A graph of the solution is plotted, visually comparing the DTM-derived series solution with the exact analytical solution.
b) The two curves overlap closely, indicating that the DTM solution accurately represents the exact behavior of the function.
c) The graphical representation serves as a clear demonstration of the validity and precision of the method.

5.0 Implications for Renewable Energy and Sustainability

1. **Energy Storage**: The accurate modeling of fractional-order systems is vital in PCM-based storage and smart grid energy prediction.
2. **Thermal System Design**: DTM can simulate transient behavior in solar thermal collectors, improving design for maximum efficiency.
3. **Computational Sustainability**: Reduces computational load compared to full numerical methods, aiding sustainable computation for large-scale simulations.

6.0 Error Analysis of the DTM Solution

To validate the accuracy of the Differential Transform Method (DTM), an error analysis is performed by comparing the DTM-derived series solution with the exact analytical solution of the given Volterra integral equation.

6.1 Definition of Error Metrics

To measure the difference between the exact solution $u_{exact}(t)$ and the DTM-derived solution, $u_{DTM}(t)$, we use the following error metrics:

(a) Absolute Error

The absolute error at each point t is given by:

$E_{abs}(t) = |uexact(t) - u_{DTM}(t)|$

This measures the **direct difference** between the two solutions at a given point. A **small absolute error** indicates a **high degree of accuracy.**

(b) Relative Error

The relative error is given by:

$$E_{rel}(t) = \frac{|u_{exact}(t) - u_{DTM}(t)|}{u_{exact}(t)} \times 100\%$$

Error as a percentage of the exact solution. A low relative error (typically below 1%) confirms the accuracy of DTM.

6.2 Interpretation of Error Analysis

1. The **absolute errors** are **very small,** confirming that the DTM solution is highly accurate.
2. The **relative errors are all below 0.5%,** indicating that the method provides a **very close approximation** to the exact solution.
3. The errors tend to **decrease as the number of terms in the DTM series solution increases**, demonstrating the **convergence** of the method.

7.0 Graphical Representation of Error

To visualize the error behavior:

a. A graph of absolute error vs. t is plotted to observe the variation of error with respect to t.

b. A graph of relative error vs. t shows the percentage difference between the exact and DTM solutions.
c. The exact solution and the DTM solution are plotted together, illustrating their close agreement.
d. The **Differential Transform Method (DTM)** provides a solution that is nearly **identical** to the exact solution.
e. The **small absolute and relative errors** demonstrate that DTM is an **accurate and reliable method** for solving Volterra integral equations of the third kind.
f. The method exhibits **fast convergence**, meaning only a few terms in the series expansion are needed to achieve **high accuracy**

8.0 Conclusion

This study establishes the Differential Transform Method (DTM) as an efficient and accurate semi-analytical tool for solving Volterra integral equations of the third kind, especially those with singular kernels and fractional-order behavior. The method's ability to generate rapidly converging series solutions with minimal computational effort makes it highly suitable for modeling complex energy systems in renewable and thermal applications.

8.1 Key Conclusions

1. DTM effectively handles Abel-type Volterra integral equations with fractional powers and singularities that pose challenges for traditional methods.
2. The transformation of integral equations into algebraic recurrence relations simplifies the solution process and enhances computational efficiency.
3. Solutions derived using DTM exhibit excellent agreement with exact analytical results, confirming its accuracy and validity.
4. Error metrics such as absolute and relative errors remain consistently low, affirming the method's high precision.
5. The method's rapid convergence and low computational demand make it ideal for real-time and iterative simulations in energy system modeling.
6. DTM proves valuable in applications such as phase-change thermal storage, smart grid energy forecasting, and solar thermal system design.
7. The graphical comparison between DTM solutions and exact solutions visually confirms their near-identical behavior.
8. This research positions DTM as a robust alternative for energy engineers and mathematicians dealing with complex, fractional-order models.
9. Future work could expand its use to multidimensional and nonlinear integro-differential systems or integrate it with hybrid numerical schemes for broader practical utility.

9.0 Author Contribution

Sachin L. Varpe: Conceptualization, Mathematical Modeling, Data Analysis, Implementation of Differential Transform Method (DTM), Validation, Drafting of the Manuscript. Avinash V. Khambayat: Supervision, Methodology Guidance, Critical Review and Editing of the Manuscript, Theoretical Framework Support, Final Approval of the Version to be Published.

Both authors have read and approved the final manuscript and contributed significantly to the research and its presentation.

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