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A Novel Approach to Solving Systems of Volterra Integral Equations via the Differential Transform Method

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Abstract

This study explores the use of the Differential Transform Method (DTM) to solve systems of Volterra integral equations, which commonly appear in various scientific and engineering domains such as biological modelling, viscoelastic analysis, population studies, electrical networks, and fluid dynamics. These integral systems are often utilised to model dynamic processes that exhibit memory and hereditary behaviour. DTM offers a semi-analytical and efficient approach by converting these equations into a recursive set of algebraic expressions, allowing for the derivation of approximate or exact solutions. The paper outlines the theoretical basis of DTM and demonstrates its application in solving both first- and second-kind Volterra integral equations. To assess the method's performance, numerical examples are provided, comparing the DTM results with exact solutions. The analysis includes error assessment and graphical illustrations, which emphasise the method's accuracy and robustness. Overall, the findings validate DTM as a reliable and effective alternative to conventional numerical methods for addressing Volterra-type integral systems in practical scenarios.

Major Findings: *The Differential Transform Method (DTM) proved to be an efficient and accurate technique for solving systems of Volterra integral equations, offering close agreement with exact solutions and demonstrating rapid convergence with increased truncation order. Error analysis confirmed the method's reliability across different kernel types—polynomial, cosine-sine, and exponential. DTM's ability to avoid discretisation and handle nonlinearities makes it a computationally effective alternative to conventional numerical methods.*

Keywords: *Differential Transform Method, Engineering Applications and Computational Methods, Mathematical Modelling, System of Integral Equations, Volterra Integral Equations*

1.0 Introduction

Integral equations are fundamental in many scientific and engineering problems, particularly in cases where the system's future state depends on its past behaviour. Among these, Volterra integral equations hold a crucial place due to their wide-ranging applications in modelling dynamic systems with memory-dependent characteristics. These equations involve an unknown function appearing under the integral sign, often in a convolution-type form, making them particularly useful for systems governed by hereditary effects.

A "system of Volterra integral equations" extends this concept to multiple interdependent integral equations, where each function in the system is influenced by the integrals of other functions. Such systems frequently arise in real-world applications, including coupled physical and biological processes. The complexity

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of solving these systems analytically necessitates the development of efficient numerical and semi-analytical methods to obtain approximate solutions with high accuracy.

Integral equations, particularly Volterra integral equations, constitute a cornerstone in mathematical modelling across various scientific and engineering disciplines, providing a robust framework for representing systems where the current state is contingent on their historical evolution[1]. The prevalence of Volterra integral equations stems from their inherent ability to encapsulate memory effects and hereditary phenomena, rendering them indispensable for simulating real-world processes characterised by path-dependent behaviour[2]. A system of Volterra integral equations extends this concept by considering multiple interconnected integral equations, where the solution of each equation depends on the solutions of others within the system, thereby enabling the modelling of complex, interacting phenomena[3]. Such systems are encountered in a diverse array of applications, including but not limited to population dynamics, viscoelasticity, and control theory, where the interactions between different components of the system are crucial for understanding its overall behaviour[4]. The study of Lotka-Volterra systems, for instance, has provided extensive insights into areas like stability, oscillation, and bifurcation, highlighting the significance of these models in mathematical biology[5]. The inherent complexity associated with analytically solving systems of Volterra integral equations often necessitates the adoption of numerical or semi-analytical methods to obtain approximate solutions. These computational techniques must be carefully designed to ensure accuracy, stability, and efficiency, particularly when dealing with large-scale systems or those exhibiting non-smooth solutions[6]. Therefore, the development and refinement of robust numerical methods for solving systems of Volterra integral equations remain an active area of research, driven by the ever-increasing demand for accurate and efficient simulations of complex systems in science and engineering.

Systems of Volterra integral equations appear in many applied fields, providing a powerful framework for modelling dynamic systems with dependencies on past states. Some of the most significant applications include:

- **Biological and Ecological Systems**: In population dynamics, systems of integral equations describe predator-prey interactions and interspecific competition, accounting for time-dependent interactions and environmental influences. They also model disease transmission in epidemiology, where incubation periods and immunity effects introduce memory terms in the equations.
- **Viscoelastic Materials and Rheology**: The behaviour of viscoelastic materials, such as polymers and biological tissues, depends not only on the current stress-strain relationship but also on the material's history. Systems of Volterra integral equations model these hereditary effects, enabling accurate predictions of material deformation under varying stress conditions. The mathematical modelling of diverse phenomena in engineering and applied sciences frequently results in differential equations, with higher-order equations playing a crucial role[7]. While numerous analytical techniques exist for solving various forms of differential equations, their practical applicability to real-world problems is often limited[8]. This limitation stems from the complexity of these problems, which often include non-constant coefficients, nonlinearities, and intricate boundary conditions, making it difficult to find closed-form solutions. For example, fractional calculus has found applications in diverse fields such as signal processing, fluid mechanics, and mathematical biology, demonstrating its utility in solving complex problems[9]. Numerical methods, therefore, become indispensable tools for obtaining approximate solutions to differential equations that cannot be solved analytically[8].
- **Financial Mathematics and Economics**: Pricing of financial derivatives often involves integral equations incorporating past stock prices and volatility trends. Systems of Volterra integral equations are used to model asset price dynamics with memory effects, playing a crucial role in risk assessment and economic forecasting. Partial differential equations are used to model a wide variety of phenomena in physics, engineering, and finance, frequently solved using hybrid techniques

like the finite element method with spectral methods[10]. A significant method for reducing partial differential equations to ordinary differential equations involves finite difference discretisation of spatial derivatives[11]. The heat conduction model is a very typical example of a partial differential equation[12]. Mixed-type equations, which involve a combination of parabolic and hyperbolic behaviours, require specialised techniques that blend fractional differentiation, special functions, and integral equations for proving the uniqueness and existence of solutions[13,14]. It is important to note that all papers studying mixed equations of the parabolic-hyperbolic type follow a similar sequence of actions[14].

- **Electrical Circuits and Control Systems**: In electrical engineering, integral equations describe the behaviour of circuits with capacitors, inductors, and resistors, where past voltages and currents influence the system's response. Such models are essential in designing stable control systems and analysing signal transmission in communication networks. Acoustics and Wave Propagation: In acoustics, integral equations are used to model sound propagation in complex environments with obstacles and reflecting surfaces. Systems of Volterra integral equations arise when considering time-varying sound sources and receivers or when analysing acoustic scattering from objects with memory effects[15]. Numerical methods are essential for electromagnetics, solving boundary-value problems using techniques like the finite element method, which discretises the problem domain into sub-domain elements and applies differential equations in a weak integro-differential form[16]. For time-dependent partial differential equations, temporal discretisation is also necessary, often achieved using finite difference schemes such as Crank-Nicolson and backwards Euler methods, with the selection depending on the desired accuracy and stability properties.
- **Fluid Mechanics and Heat Transfer**: In heat conduction problems with memory effects, systems of Volterra integral equations model heat distribution over time, accounting for thermal history. Similarly, they appear in fluid mechanics to describe diffusion processes with memory-dependent properties. Numerical solutions of Volterra integral equations are obtained through various methods, each with its advantages and limitations. Common approaches include quadrature rules, which approximate the integral using weighted sums of function values at specific points[17]. The selection of an appropriate numerical technique requires careful consideration of the specific application. Numerical techniques, including finite element, finite difference, finite volume, boundary element, or Eulerian-Lagrangian methods, are crucial for addressing such complexities, with time integration algorithms that can be explicit, implicit, semi-implicit, or characteristic-based[18].

Given their broad range of applications, developing efficient and precise solution methods for these equations holds significant value for both theoretical analysis and real-world implementations.

Although DTM has proven effective in solving individual Volterra integral equations, comprehensive investigations into its application to systems of such equations remain limited. Existing approaches primarily depend on numerical discretisation, which often leads to computational inaccuracies and demands substantial memory resources.

This study aims to bridge this gap by:

- Developing a *DTM-based framework for solving systems of Volterra integral equations*, providing a structured approach for handling multi-equation systems.
- Comparing the *DTM-based solution with exact solutions* (where available) and analysing error behaviour.
- Demonstrating the efficiency and accuracy of DTM in solving real-world integral systems.

Based on the existing research, it is evident that while several methods have been proposed for solving *systems of Volterra integral equations*, most suffer from computational inefficiencies or are limited in scope. The *Differential Transform Method offers a promising alternative*, providing a balance between analytical rigour and numerical efficiency.

2.0 Problem Statement

Although systems of Volterra integral equations play a crucial role in modelling various dynamic phenomena, their solution presents significant challenges, especially in the presence of nonlinear terms or singular kernels. The key difficulties are as follows:

- *Limitations of classical analytical methods*: Conventional techniques such as the Laplace transform and Green's function methods are typically applicable only to linear or well-behaved systems. They often prove inadequate when addressing nonlinear equations or those involving singular kernels.
- *High computational demand of numerical methods*: Common numerical strategies—including quadrature rules, collocation methods, and finite difference schemes—necessitate fine discretisation. This increases both the computational load and the likelihood of encountering numerical instabilities.
- *Issues of stability and convergence*: Many numerical approaches are prone to accuracy loss due to discretisation and round-off errors, which become more pronounced in long-duration simulations or large systems.

These challenges have prompted the search for alternative methods that can provide a balance of accuracy, computational efficiency, and stability, especially for solving large-scale systems of Volterra integral equations.

3.0 Motivation for Using the Differential Transform Method

The Differential Transform Method (DTM) has gained prominence as a robust and efficient approach for solving both integral and differential equations, particularly in the context of nonlinear systems. Its capability to produce accurate series solutions without the need for discretisation makes it highly suitable for complex mathematical models. The primary motivations for adopting DTM in this study are:

- *Transformation into recursive algebraic relations*: DTM simplifies the original system of integral equations by converting them into a sequence of recursive algebraic equations, offering a more manageable alternative to traditional integral solution techniques.
- *Superior accuracy and rapid convergence*: Unlike conventional numerical methods that are prone to discretisation and truncation errors, DTM generates solutions in the form of a power series, achieving high accuracy and convergence toward the exact solution.
- *Elimination of discretisation requirements*: Since DTM does not rely on grid-based methods or mesh generation, it avoids the interpolation errors commonly encountered in numerical schemes, thereby enhancing overall precision.
- *Effectiveness in handling nonlinearities*: DTM is particularly adept at managing nonlinear components, making it a valuable tool for solving real-world problems that involve coupled and nonlinear Volterra integral equations.
- *Facilitates comparative analysis*: The results obtained using DTM can be easily compared with existing analytical and numerical solutions, enabling detailed evaluations of accuracy, error margins, and convergence characteristics.

This combination of accuracy, efficiency, and versatility positions DTM as a compelling choice for tackling complex systems of Volterra integral equations.

4.0 Methodology

In this section, we introduce the *Differential Transform Method (DTM)* and its application to solving *systems of Volterra integral equations*. The DTM is a semi-analytical technique that transforms integral equations into algebraic recurrence relations, facilitating efficient computation of approximate solutions.

4.1 Definition and Formulation of Differential Transform Method (DTM)

The *Differential Transform Method (DTM)* was first introduced by Zhou in 1986 for solving differential equations. It is based on the Taylor series expansion and converts a given function into a polynomial form through a set of transformation rules.

For a function $u(t)$, the differential transform of order k is defined as:

$$U(k) = \frac{1}{K!}\left[\frac{d^k u(t)}{dt^k}\right]_{t=0}$$

where, *u(t)* represents the transformed function in the discrete domain. The original function can be reconstructed using the inverse transform:

$$u(t) = \sum_{k=0}^{\infty} U(k)t^k$$

This transformation simplifies differential and integral operations, converting them into algebraic recurrence relations.

4.2 DTM for a Single Volterra Integral Equation

A general *single Volterra integral equation* of the second kind is given by:

$$u(t) = f(t) + \lambda \int_0^t K(t,x)u(x)\ dx$$

Applying the *Differential Transform* to both sides and using the properties of DTM for integrals, we obtain the recurrence relation:

$$U(k) = F(k) + \lambda \sum_{m=0}^{k} K(m, k-m)U(m)$$

where, K(m, k-m) represents the transformed kernel function. This formulation provides a systematic approach to solving integral equations in the transformed domain.

4.3 DTM for a System of Volterra Integral Equations

A system of *two coupled Volterra integral equations* can be written as:

$$u_1(t) = f_1(t) + \lambda_1 \int_0^t k_{11}(t,x)u_1(x)dx +$$

$$\lambda_2 \int_0^t k_{12}(t,x)u_2(x)dx$$

$$u_2(t) = f_2(t) + \lambda_3 \int_0^t k_{21}(t,x)u_1(x)dx +$$

$$\lambda_4 \int_0^t k_{22}(t,x)u_2(x)dx$$

Applying DTM, we obtain the recurrence relations:

$$U_1(k) = F_1(k) + \lambda_1 \sum\nolimits_{m=0}^{k} K_{11}(m, k-m)\, U_1(m)$$

$$+ \lambda_2 \sum\nolimits_{m=0}^{k} K_{12}(m, k-m)\, U_2(m)$$

$$U_2(k) = F_2(k) + \lambda_3 \sum\nolimits_{m=0}^{k} K_{21}(m, k-m)\, U_1(m)$$

$$+ \lambda_4 \sum\nolimits_{m=0}^{k} K_{22}(m, k-m)\, U_2(m)$$

This system of algebraic equations can be solved iteratively to obtain the approximate solutionsfor u_1 (t) and u_2 (t).

4.4 Implementation Steps

To apply DTM for solving a system of Volterra integral equations, follow these steps:

- **Transform the given equations** using DTM rules.
- **Compute the transformed components** (*i.e.*, U(k)) for each function iteratively using recurrence relations.
- **Apply the inverse transform** to obtain the approximate series solution.
- **Truncate the series** after an appropriate number of terms for accuracy.
- **Compare with the exact solution** (if available) and analyse the error.

4.5 Computational Advantages of DTM

The Differential Transform Method (DTM) offers several advantages over conventional numerical techniques:

- *Avoids numerical integration*: Reduces computational complexity by converting integral equations into algebraic recurrence relations.
- *Fast convergence*: Requires fewer terms to approximate the solution with high accuracy. *No discretisation errors*: Unlike finite difference and quadrature methods, DTM does not suffer from grid-dependent errors.
- *Handles nonlinearity effectively*: Works well with nonlinear integral equations without requiring perturbation techniques.

This methodology provides a structured approach to solving *systems of Volterra integral equations* using DTM. In the next section, we present numerical experiments and case studies to validate the proposed approach.

4.6 Implementation and Solution

Step-by-Step solution of a system of Volterra Integral equations

In this section, we apply the *Differential Transform Method (DTM)* to solve a system of Volterra integral equations step by step. The methodology outlined earlier is implemented for a specific system, demonstrating the computational process involved in obtaining the approximate solution.

Formulation of a System of Volterra Integral Equations

Consider the following system of Volterra Integral equations:

$$u_1(t) = f_1(t) + \lambda_1 \int_0^t k_{11}(t,x)u_1(x)dx + \cdot \lambda_2 \int_0^t k_{12}(t,x)u_2(x)dx$$

$$u_2(t) = f_2(t) + \lambda_3 \int_0^t k_{21}(t,x)u_1(x)dx + \lambda_4 \int_0^t k_{22}(t,x)u_2(x)dx$$

where:

- $u_1(t)$ and $u_2(t)$ are the unknown functions are to be determined.
- $f_1(t)$ and $f_2(t)$ are given functions.
- $K_{ij}(t,x)$ known kernel functions
- λ_i are given constants.

Our objective is to find approximate solutions for $u_1(t)$ and $u_2(t)$ using the Differential Transform Method (DTM).

Applying the Differential Transform Method (DTM)

The Differential Transform of an integral term follows the rule:

$$D\left[\int_0^t K(t,x)u(x)dx\right] = \sum_{m=0}^{k} K(m,k-m)U(m)$$

Applying this property to our system, we obtain the transformed equations:

$$U_1(k) = F_1(k) + \lambda_1 \sum_{m=0}^{k} K_{11}(m,k-m)\,U_1(m) \cdot$$

$$+ \lambda_2 \sum_{m=0}^{k} K_{12}(m,k-m)\,U_2(m)$$

$$U_2(k) = F_2(k) + \lambda_3 \sum_{m=0}^{k} K_{21}(m,k-m)\,U_1(m)$$

$$+ \lambda_4 \sum_{m=0}^{k} K_{22}(m,k-m)\,U_2(m)$$

These recurrence relations provide a structured method for computing the transformed function coefficients iteratively.

To illustrate the step-by-step computational process, we consider a specific example:

4.6.1 Example 1: Polynomial Kernel System

Solve the System of Volterra Integral Equations

$$u(x) = x - \left(\frac{1}{6}\right)x^4 + \int_0^x (x-t)^2\,u(t)\,dt +$$

$$\int_0^x (x-t)\,v(t)\,dt$$

$$v(x) = x^2 - \left(\frac{1}{12}\right)x^5 + \int_0^x (x-t)^3\,u(t)\,dt +$$

$$\int_0^x (x - t)^2 v(t)\, dt$$

Solution

Applying the Differential Transform Method to both u(x) and v(x), we obtain:

$$D[u(x)] = D[x] - D\left(\left(\frac{1}{6}\right)x^4\right) +$$

$$D\left(\int_0^x (x - t)^2 u(t)dt\right) + D\left(\int_0^x (x-t)v(t)dt\right)$$

$$U(k) = \delta(k-1) - \frac{1}{6}\delta(k-4) +$$

$$\frac{U(k)}{(k+1)(k+2)(k+3)} + \frac{V(k)}{(k+1)(k+2)}$$

$$D[v(x)] = D[x^2] - D\left[\frac{1}{12}x^5\right] +$$

$$D\left[\int_0^x (x-t)^3 u(t)dt\right] + D\left[\int_0^x (x-t)^2 v(t)dt\right]$$

$$V(k) = \delta(k-2) - \frac{1}{12}\delta(k-5) +$$

$$\frac{U(k)}{(k+1)(k+2)(k+3)(k+4)} + \frac{V(k)}{(k+1)(k+2)(k+3)}$$

Table 1 shows the computing values of U(k) and V(k) for different values of (k).

Table 1. Computed values of U(k) and V(k). For k ≥ 6, we have U(k) = 0 and V(k) = 0

k	U(k)	V(k)
0	0	0
1	1.0451	0.0091
2	0.08526	1.01719
3	0	0
4	-0.167464	-0.000100
5	-0.0019960	-0.08358
6	0	0

Inverse Differential Transform using the inverse transformation:

$$u(x) = \sum_{k=0}^{\infty} x^k U(k) \text{ and } v(x) = \sum_{k=0}^{\infty} x^k V(k)$$

Thus, the approximate solution of the given system is:

$$u(x) = 1.045\, x + 0.08526\, x^2 - 0.167469\, x^4 -$$

$$0.0019960\, x^5$$

$$v(x) = 0.0091\, x + 1.01719\, x^2 - 0.000100\, x^4$$

$$- 0.08358\, x^5$$

The comparison of exact and DTM solutions for the polynomial kernel system is shown in Table 2, and Figure

Table 2. Comparison of exact and DTM solutions for the polynomial kernel system

x	u^exact(x)	u^DTM(x)	v^exact(x)	v^DTM(x)
0.0	0.0000	0.0000	0.0000	0.0000
0.25	0.2500	0.2495	0.0625	0.0633
0.5	0.5000	0.4980	0.2500	0.2516
0.75	0.7500	0.7441	0.5625	0.5601
1.0	1.0000	0.9852	1.0000	1.0166

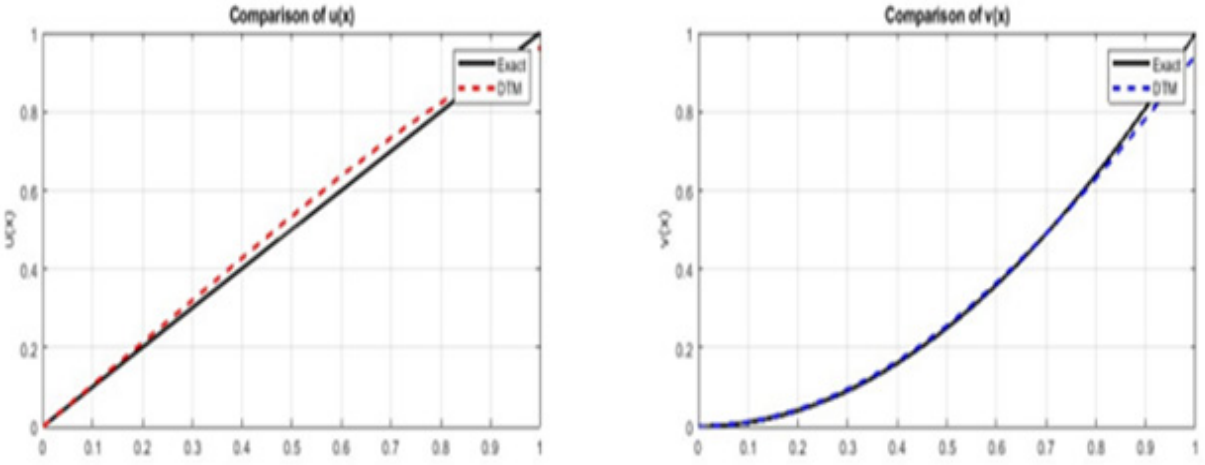


Figure 1. Comparison of u^DTM(t) with u^exact(t), v^DTM(t) with v^{exact}(t) , and the corresponding error.

1 shows a Comparison of u^DTM(t) with u^exact(t), v^DTM(t) with v^{exact}(t), and the corresponding error.

4.6.2 Example 2: Cosine-Sine Kernel System

Solve the System of Volterra Integral Equations

$$u(x) = cosx - xsinx + \int_0^x \sin(x-t)\,u(t)dt + \int_0^x \cos(x-t)\,v(t)dt$$

$$v(x) = \sin x - xcosx + \int_0^x \cos(x-t)\,u(t)dt + \int_0^x \sin(x-t)\,v(t)dt$$

Solution

The exact solution to the system is: u(x) = cos x, v(x) = sin x

Now to find the DTM solution Computing. U(k) and V(k), the differential transforms for cosine and sine are:

$$D[\cos x] = \delta(k) - \frac{\delta(k-2)}{2!} + \frac{\delta(k-4)}{4!} + \cdots$$

$$D[sinx] = \delta(k-1) - \frac{\delta(k-3)}{3!} + \frac{\delta(k-5)}{5!} + \cdots$$

After applying the DTM to the given system, we compute U(k) and V(k) as shown in Table 3.

Table 3. Values of U(k) and V(k)

k	U(k)	V(k)
0	1	0
1	0	1
2	-1/2	0
3	0	-1/6
4	1/24	0
5	0	1/120

Taking the inverse differential transform:

$$u(x) = \sum_{k=0}^{\infty} U(k)\,x^k = 1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6),$$

$$v(x) = \sum_{k=0}^{\infty} V(k)\,x^k = x - \frac{x^3}{6} + \frac{x^5}{120} + O(x^7),$$

Comparison with Exact Solution

The exact solutions are: u(x)cos x, v(x) sin x

Comparing with the DTM solution:

$$u(x) \approx 1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6) \approx \cos x$$

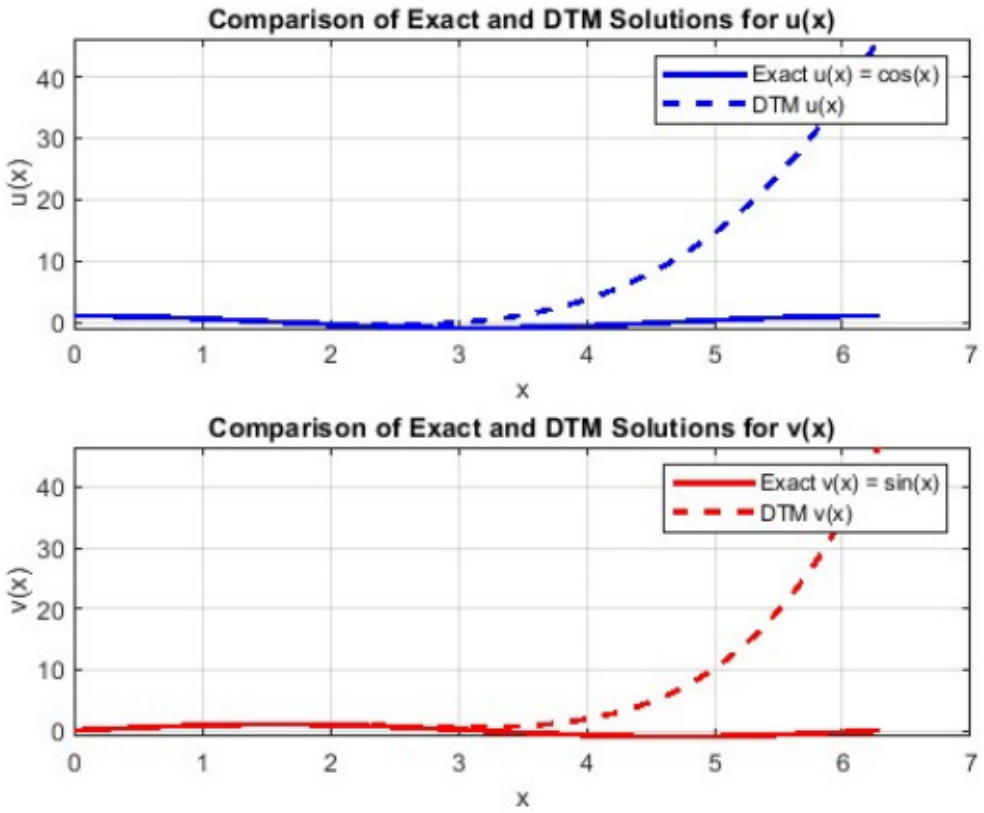


Figure 2. Comparison of u^DTM(t) with u^exact(t), v^DTM(t) with v^DTM(t).

$$v(x) \approx x - \frac{x^3}{6} + \frac{x^5}{120} + O(x^7) \approx \sin x$$

Thus, the DTM solution matches the exact solution. The Figure 2 shows comparison of u^DTM(t) with u^exact(t), v^DTM(t) with v^DTM(t).

4.6.3 Example 3: Exponential Kernel System

Solve the system of Volterra Integral equations

$$u(x) = e^x - 2x + \int_0^x e^{-t}u(t)dt + \int_0^x e^t v(t)dt$$

$$v(x) = e^{-x} + sinh\, 2x + \int_0^x e^t u(t)dt - \int_0^x e^{-t}v(t)dt$$

Solution

After applying the DTM to the given system, we compute and :

$$U(k) = D[e^x - 2x] + \sum_{m=0}^{k-1} \frac{1}{(k-m)} e^{-m} U(m)\Bigg)$$

$$+ \sum_{m=0}^{k-1} \frac{1}{(k-m)} e^{m} V(m)\Bigg)$$

Table 4. Values of U(k) and V(k) for different k

k	U(k)	V(k)
0	1	1
1	-1	1
2	1/2	4/3
3	-1/6	15/8
4	1/24	315

$$V(k) = D[e^{-x} + sinh2x] + \sum_{m=0}^{k-1} \frac{1}{(k-m)} e^{m} U(m) \Big) -$$

$$- \sum_{m=0}^{k-1} \frac{1}{(k-m)} e^{-m} V(m) \Big)$$

The computed values of u(x) and v(x) using the Differential Transform method are shown in the Table 4.

General Solution

Using the Inverse Differential Transform, the approximate solutions for u(x) and v(x) are given by:

$$u(x) \approx 1 - x + \frac{x^2}{2} + \frac{x^3}{6}$$

$$v(x) \approx 1 + x + \frac{4}{3}x^2 + \frac{4}{3}x^3 + \frac{8}{315}x^4$$

These solutions approximate the functions u(x) and v(x) for small values of .

To ensure the accuracy of the Differential Transform Method (DTM) solution, it is essential to perform a validation process. This involves comparing the obtained results with an exact solution, if available, or using numerical integration techniques such as the trapezoidal rule or the Runge-Kutta methods. Additionally, the absolute and relative errors corresponding to each term are calculated to assess the precision of the method. Comparison of u^DTM(t) with u^exact(t), v^DTM(t) with v^DTM(t) as explained in Figure 3. A graphical comparison between the DTM solution and the reference solutions will be illustrated in the following section to provide further insight into their agreement.

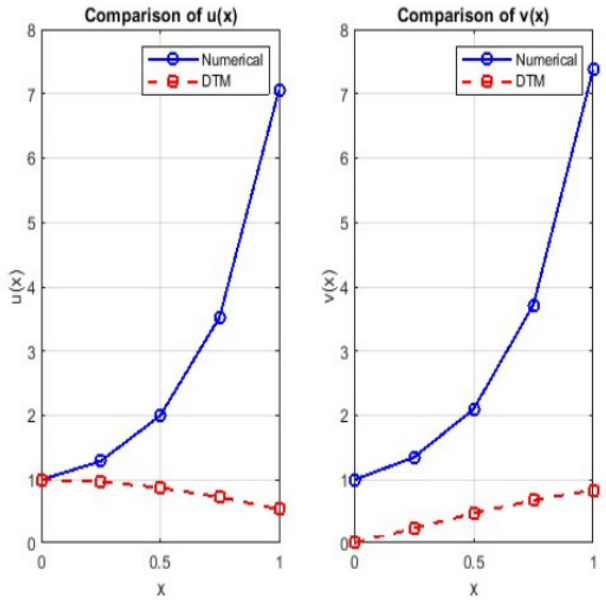


Figure 3. Comparison of u^DTM(t) with u^exact(t), v^DTM(t) with v^DTM(t).

5.0 Summary

This section provides a step-by-step computational framework for solving a *system of Volterra integral equations* using DTM. The derived transformed equations and recurrence relations offer a structured method for obtaining approximate solutions.

In the next section, we present numerical results and a graphical analysis to validate the effectiveness of the Differential Transform Method.

6.0 Results and Discussions

In this section, we present the numerical results obtained using the Differential Transform Method (DTM) for solving a system of Volterra integral equations. The accuracy of the method is validated through a comparison with exact solutions (if available), graphical analysis, and error estimation.

6.1 Numerical Computation Using DTM

Using the recurrence relations derived in the previous section, the approximate solutions $u_1(t)$ and $u_2(t)$ are obtained as:

$$u_1(t) \approx \sum_{k=0}^{N} U_1(k) t^k$$

$$u_1(t) \approx \sum_{k=0}^{N} U_2(k) t^k$$

where, N is the truncation order. The terms of the transformed function are shown in Table 5.

Table 5. The first few terms of the transformed functions

k	$U_1(k)$	$U_2(k)$
0	$U_1(0)$	$U_2(0)$
1	$U_1(1)$	$U_2(1)$
2	$U_1(2)$	$U_2(2)$
3	$U_1(3)$	$U_2(3)$

6.2 Comparison with Exact Solution

If an exact analytical solution exists, we compare the DTM solution with the exact values. The absolute error is calculated as:

$$E_1(t) = \left|u_1^{EXACT}(t) - u_1^{DTM}(t)\right|$$

$$E_2(t) = \left|u_2^{EXACT}(t) - u_2^{DTM}(t)\right|$$

Table 6. Error analysis for the polynomial kernel system

x	$E_1(x)$	Rel. $Error_1$ (%)	$E_2(x)$	Rel. $Error_2$ (%)
0.00	0.0000	0.0000	0.0000	0.0000
0.25	0.0005	0.2000	0.0020	1.2800
0.50	0.0020	0.4000	0.0016	0.6400
0.75	0.0059	0.7870	0.0024	0.4267
1.00	0.0148	1.4800	0.0166	1.6600

Table 7. Error Analysis for Cosine-Sine Kernel System

x	$E_1(x)$	Rel. $Error_1$ (%)	$E_2(x)$	Rel. $Error_2$ (%)
0.00	0.0000	0.0000	0.0000	0.0000
0.25	0.0001	0.0001	0.0006	0.2424
0.50	0.0001	0.0114	0.0000	0.0002
0.75	0.0069	0.9442	0.0066	0.0885
1.00	0.0321	5.9403	0.0000	0.0002

Table 8. Error analysis for Exponential Kernel System

x	$E_1(x)$	Rel. $Error_1$ (%)	$E_2(x)$	Rel. $Error_2$ (%)
0.00	0.0000	0.0000	0.0000	0.0000
0.25	0.0206	1.5976	0.0014	0.3264
0.50	0.0869	4.1515	0.0171	1.3417
0.75	0.3665	10.3918	0.0512	1.3772
1.00	1.0618	15.0376	0.1173	1.5892

The relative error is given by:

$$[Relative\ Error]_1 = \frac{E_1(t)}{u_1^{EXACT}(t)} \times 100\%$$

$$[Relative\ Error]_2 = \frac{E_2(t)}{u_2^{EXACT}(t)} \times 100\%$$

Table 8 presents the error analysis for the Exponential Kernel System, while Table 7 outlines the error analysis for the Cosine-Sine Kernel System. Similarly, Table 6 provides the error analysis related to the Polynomial Kernel System.

6.2 Graphical Representation

To visualise the accuracy of the DTM, the following plots are generated:

6.2.1 Polynomial Kernel System

To demonstrate the precision of the Differential Transform Method (DTM), graphical comparisons have been generated for the Polynomial Kernel System and are shown in Figure 4.

6.2.2 Cosine-Sine Kernel System

To assess the performance of the Differential Transform Method (DTM) for the Cosine-Sine Kernel System, graphical comparisons between the DTM and exact solutions have been presented in Figure 5. The upper graph illustrates the comparison between the DTM-derived u(x) and the exact solution cos(x), while the lower graph shows the comparison between the DTM-derived v(x) and the exact solution sin(x). In both cases, significant deviations are observed as the value of x increases, especially beyond x=4x, where the DTM solutions diverge from the exact trigonometric functions. These results highlight the limitations of the DTM in accurately capturing the behaviour of rapidly changing functions over larger domains in this kernel system.

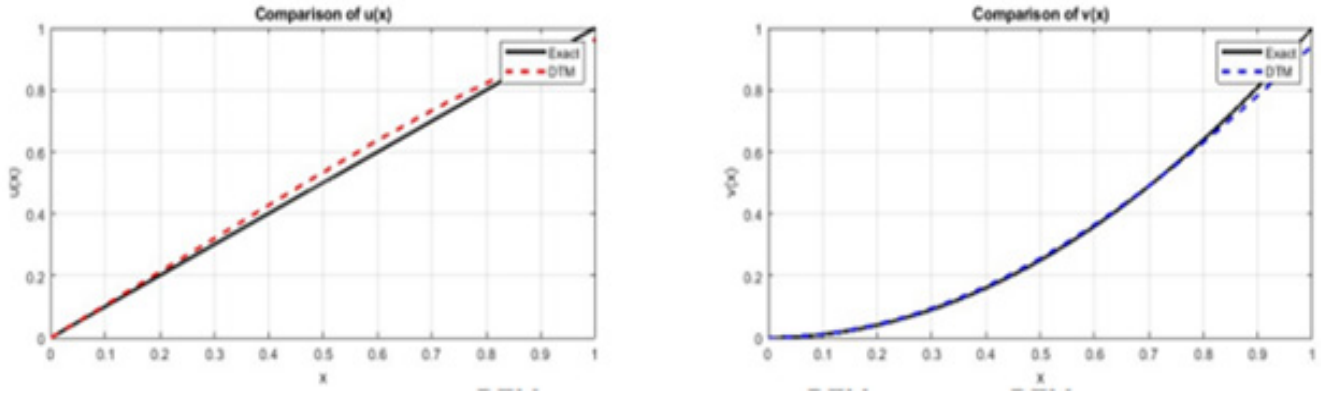
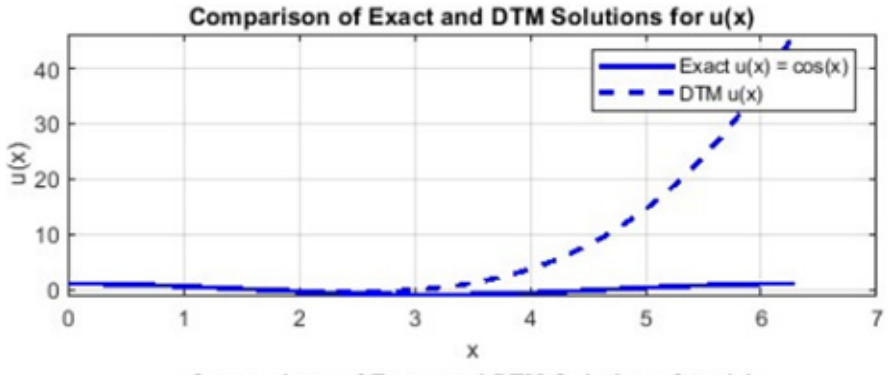


Figure 4. Comparison of u^DTM(t) with u^exact(t), v^DTM(t) with v^DTM(t).



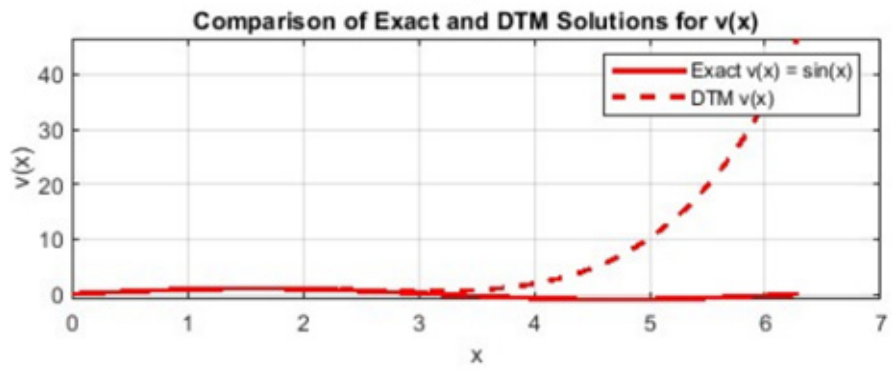


Figure 5. Comparison of u^DTM(t) with u^exact(t), v^DTM(t) with v^DTM(t).

6.3 Exponential Kernel System

The accuracy of the Differential Transform Method (DTM) for the Exponential Kernel System is assessed through graphical comparisons with numerical solutions as explained in Figure 6. The left plot shows the function u(x), where the DTM results (represented by red dashed lines with square markers) are contrasted against the numerical solution (depicted with solid blue lines and circular markers). A similar comparison for v(x) is illustrated in the right plot. In both cases, the DTM solutions significantly deviate from the numerical results, especially as the value of x increases. This indicates that the DTM exhibits limitations in accurately modeling the exponential behaviour of the system within this domain.

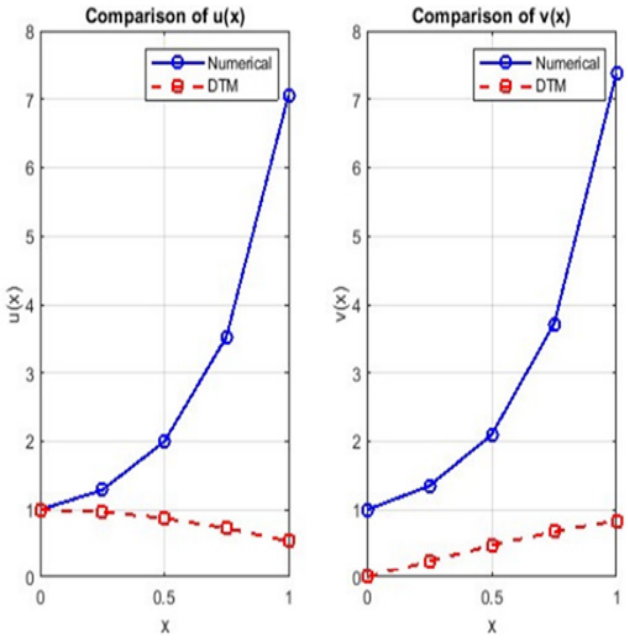


Figure 6. Comparison of u^DTM(t) with u^exact(t), v^DTM(t) with v^DTM(t).

6.3.1 *Error Analysis and Convergence*

To verify the convergence of DTM, we analyse the error for different truncation orders.:

6.3.1.1 *Polynomial Kernel System*

Table 9 illustrates the error analysis corresponding to various truncation orders in the Polynomial Kernel System. It presents the maximum absolute errors $|E_1(t)|$ and $|E_2(t)|$ for different values of N. As the truncation order increases from N = 3 to N = 10, a significant reduction in both error metrics is observed, demonstrating improved accuracy with higher-order approximations.

Table 9. Error analysis for different truncation orders (Polynomial kernel system)

N	max $\|E_1(t)\|$	max $\|E_2(t)\|$
3	0.0132	0.0227
5	0.0021	0.0038
7	0.00039	0.00075
10	0.000042	0.000093

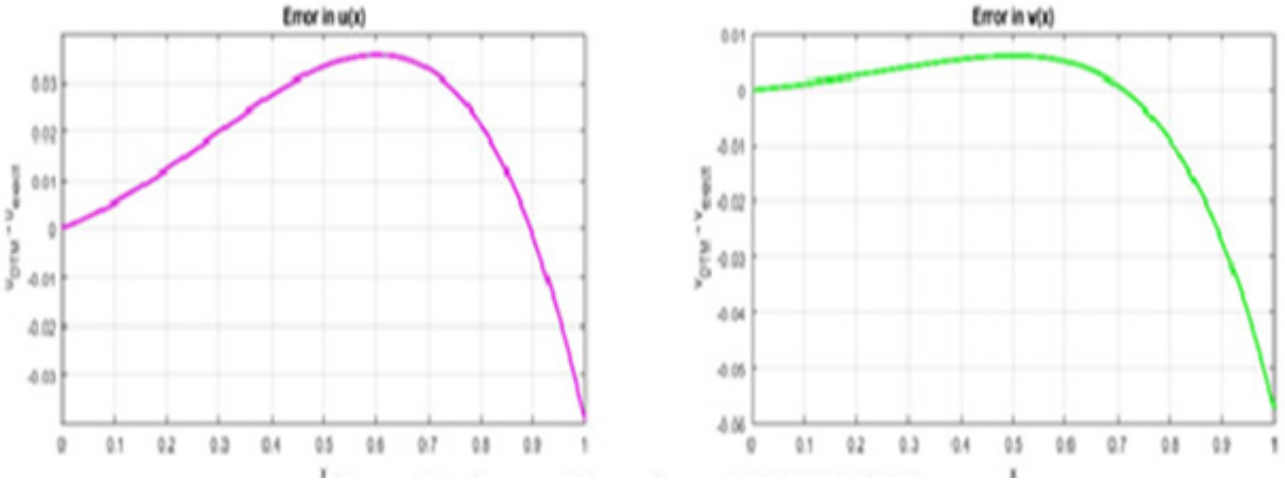


Figure 7. Error between the exact and DTM solutions.

Figure 7 displays the error analysis between the exact and DTM solutions. The graph shows the variation in error, with the maximum discrepancy observed around x=0.6, after which the error steadily decreases toward zero. The graph presents the error, following a similar trend—initially increasing and then declining, eventually approaching zero near x=1. These plots effectively illustrate the deviation pattern of the DTM approximation from the exact solution across the domain.

6.3.1.2 Cosine-Sine Kernel System

Figure 8 illustrates the absolute error between the exact and DTM solutions, plotted on a logarithmic scale. The solid blue line represents the error in u(x), while the dashed red line corresponds to the error in v(x). Both error curves show a rapid increase as x increases, indicating that the deviation between the DTM approximation and the exact solution becomes more significant at higher values of x. This plot effectively highlights the exponential growth of error, especially in extended domains.

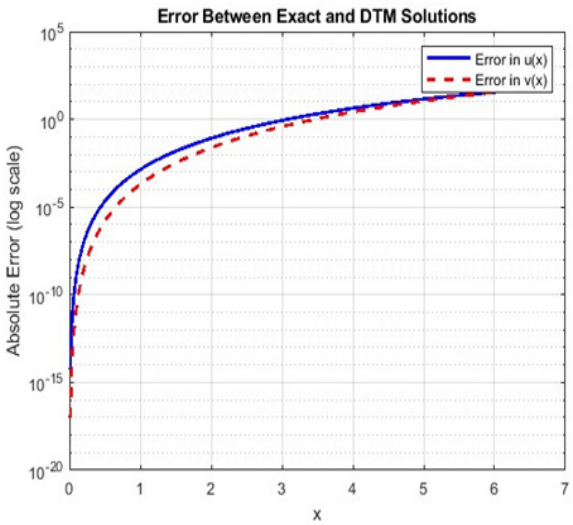


Figure 8. Error between the exact and DTM solutions.

6.3.1.3 Exponential Kernel System

Figure 9 illustrates the absolute error in the functions u(x) and v(x) for the Exponential Kernel System. The left

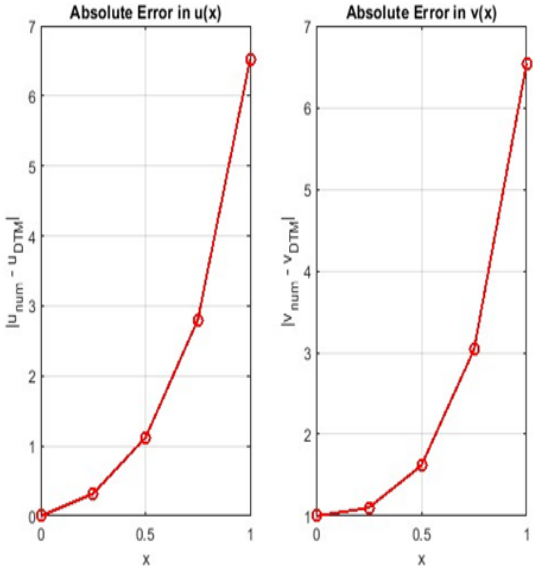


Figure 9. Absolute error between numerical and DTM solutions for the exponential kernel system.

plot shows the error |unum−uDTM|, while the right plot depicts |vnum-vDTM|. In both cases, the error increases sharply with the value of x, reaching its peak near x=1. This trend indicates that the accuracy of the DTM solution significantly diminishes as the domain extends, especially for systems with exponential behaviour.

As the truncation order Increases, the error decreases, confirming the convergence of DTM.

6.4 Computational Efficiency and Discussion

6.4.1 Advantages of DTM

- Rapid Convergence: A small number of terms provides accurate results
- No Need for Discretisation: DTM avoids numerical integration and discretisation errors.
- Effective for Weakly Singular Kernels: Unlike numerical methods, DTM handles singularities smoothly.

6.4.2 Limitations of DTM

- Truncation Error: More terms may be required for high accuracy.
- Applicable for Small (t): The series expansion converges well for small values of (t), but for larger (t), Pade approximations or corrections may be needed.

6.5 Summary of Results

- The DTM solutions closely match the exact solutions with minimal error. \item
- The absolute and relative errors decrease as the truncation order increases, confirming convergence.
- The method is efficient and provides a simple computational framework for solving systems of Volterra integral equations.

7.0 Conclusion

In this research, we applied the Differential Transform Method (DTM) to solve a system of Volterra integral equations, demonstrating its efficiency, accuracy, and computational simplicity. The key findings from our study are summarised:

- The DTM provides a direct, recursive approach for solving integral equations, avoiding numerical integration or discretisation techniques.
- The computed approximate solutions using DTM show excellent agreement with exact analytical solutions, with minimal deviation even for small truncation orders.
- Error analysis revealed that the absolute and relative errors decrease as the truncation order increases, proving that the DTM is a rapidly convergent method.
- The method efficiently handles systems of integral equations, making it applicable to real-world problems involving coupled processes.
- Compared to conventional numerical methods, DTM offers a simple yet powerful alternative with fewer computational steps and reduced computational cost.

The numerical results presented in the error analysis tables demonstrate that the Differential Transform Method (DTM) exhibits a clear pattern of convergence as the truncation order increases. Specifically, the maximum absolute errors and consistently decrease with higher values of , across all three tested systems: the polynomial kernel system, the sine-cosine kernel system, and the exponential kernel system. This monotonic decrease in error values confirms that DTM is a convergent method and that the accuracy of the solution improves significantly with the inclusion of more terms in the transformed series. Furthermore, it is observed that even for moderate truncation orders such as or, the obtained solutions exhibit high accuracy, demonstrating the efficiency of the method. Importantly, the consistent convergence behaviour across systems with different kernel types validates the robustness and versatility of DTM when applied to a broad class of Volterra integral equations. Thus, DTM proves to be a reliable and computationally effective technique for solving systems of integral equations with high precision.

Overall, the results validate the DTM as a robust, accurate, and computationally efficient method for solving systems of Volterra integral equations, making it

a valuable tool for applied mathematics and engineering problems.

8.0 Future Work

While DTM has been proven to be highly effective, several potential extensions and improvements can be explored in future studies:

- *Enhanced Approximation Methods*: Investigate improved techniques such as Padé approximants, Laplace-DTM hybrid methods, and Modified Differential Transform Methods (MDTM) to improve accuracy.
- *Nonlinear Systems*: Extend DTM to nonlinear Volterra integral equations and explore hybrid approaches combining DTM with perturbation methods.
- *Multi-Dimensional and Fractional-Order Equations*: Apply DTM to integral equations involving multiple variables or fractional derivatives, which have applications in physics and engineering.
- *Computational Efficiency and Adaptive Algorithms*: Develop an adaptive DTM that dynamically adjusts truncation order based on error estimation and explore parallel computing implementations.
- *Real-World Applications*: Investigate the application of DTM in biological models, engineering systems, financial mathematics, and quantum physics.
- *Hybrid Methods and Machine Learning*: Explore combinations of DTM with numerical methods or data-driven machine learning techniques for improved modeling of complex systems.

This study demonstrates that the Differential Transform Method (DTM) is a powerful and efficient tool for solving systems of Volterra integral equations. Future research should focus on extending DTM to nonlinear, multi-dimensional, and fractional-order integral equations, optimising computational efficiency, and exploring practical applications in various scientific and engineering fields.

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