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SOME PROPERTIES OF SCHUBERT VARIETIES

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Introduction. This paper is a continuation of our earlier paper [**10**]. With notations as in [**10**] (some of which are recalled in 1 and 2, below), we prove the following

1. THEOREM (6.1, *below*). *The Picard group of a non-trivial Schubert variety* Ω *in the Grassmann variety* $G_{d,n}$ *is* $\mathbf{Z}$ *and is generated by the class of* $\mathcal{O}_\Omega(1)$.

In fact, our method of proof of the vanishing theorems for Ω (and of the above theorem) yields the following

2. THEOREM. *Let* $W \subseteq G_{d,n}$ *be an equidimensional closed subscheme whose irreducible components are Schubert varieties. Then* (*i*) W *is reduced* (*ii*) *Vanishing theorems hold for* W (*iii*) *Postulation formula holds for* W (*iv*) W *is arithmetically Cohen-Macaulay and* (*v*) *Pic* $W = \mathbf{Z}$ *if dim* $W \geqslant 1$.

We then establish the precise relationship that exists between determinental loci and Schubert varieties (cf. 3 and 7, below). We prove the

3. THEOREM. (7.3, *below*). *Determinantal loci are canonically isomorphic to the scheme-theoretic intersection of Schubert varieties with the big open cell in the opposite cellular decomposition of* $G_{d,n}$.

As an obvious consequence of this theorem, it follows that the determinantal loci are integral, Cohen-Macaulay and normal. A useful fact to be noted is that Schubert varieties are cones in a neighbourhood of their distinguished point, namely the point Schubert variety (cf. Remark 7.4 (1 and 3), below). With this observation we prove the following.

4. THEOREM (8.1, *below*). *Let* Ω *be a Schubert variety in* $G_{d,n}$ *and* D *be the determinantal locus in* Ω. *Then the following statements are*

equivalent: (*i*) Ω *is arithmetically factorial* (*ii*) *there exists a unique Schubert variety of codimension* 1 *in* Ω (*iii*) Ω *is non-singular* (*in fact, isomorphic to a Grassmann variety*) (*iv*) Ω *is factorial* (*v*) D *is factorial* (*at its vertex*).

It is easy to recognise the non-singular Schubert varieties in $G_{d,n}$. By a result of Murthy ([9], p. 419), the non-singular Schubert varieties and hence also the codimension 1 Schubert varieties contained in them are arithmetically Gorenstein.

5. Several proofs are now available for the arithmetic (normality and) Cohen-Macaulay nature of Schubert varieties. Here we include another proof (cf. 4.1 below) in which we construct a canonical system of parameters at the vertex of $\hat{\Omega}$. This system induces also a system of parameters at the vertex of the determinantal locus that Ω contains (cf. Remark 7.4 (3), below). Finally, we point out (in 5.1, below) that the arithmetic normality of Ω is an immediate consequence of Pieri's formula.

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1. Notation and Terminology. Let V be an n-dimensional (finite) vector space over an algebraically closed field k of arbitrary characteristic. For a fixed integer d, $1 \leqslant d < n$, $G_{d,n}$ denotes the Grassmann variety of d-dimensional linear subspaces of V. Let $\mathbf{P} = \mathbf{P}(\wedge^d(V))$ denote the projective space in which $G_{d,n}$ is canonically imbedded.
Let L denote the hyper-plane bundle on $\mathbf{P}$ as well as its restriction to all subvarieties of $\mathbf{P}$. For any subvariety X of $\mathbf{P}$, $\hat{X}$ denotes the cone over X, and (0) denotes the vertex of $\hat{X}$.

We fix a celullar decomposition of $G_{d,n}$ ([**10**], 4, p. 147). The Schubert cells are parametrised by the set $S = \{(a) = (a_1, \ldots, a_d) \mid 1 \leqslant a_1 < \ldots < a_d \leqslant n\}$. For $(a) \in S$, $U_{(a)}$ denotes the Schubert cell, and $\Omega_{(a)}$ = Zariski closure of $U_{(a)}$ in $G_{d,n}$ (with the canonical reduced scheme structure) denotes the corresponding Schubert variety.

For the standard partial order on S, namely, $(a) \leqslant (b)$ if $a_i \leqslant b_i$, $1 \leqslant i \leqslant d$; we have $(a) \leqslant (b)$ if and only if $\Omega_{(a)} \supseteq \Omega_{(b)}$ ([**10**], 1, p. 151). We write $(\min) = (1, \ldots, d)$ and $(\max) = (n - d + 1, \ldots, n)$ which are the (unique) minimal and maximal elements of S. We have $\Omega_{(\min)} = G_{d,n}$ and $\Omega_{(\max)} = \{x_0\}$ the point Schubert variety.

Fix a basis $e_1, \ldots, e_n$ for V, and identify $V^d = V \oplus \ldots \oplus V$, ($d$ copies) with the set of $d \times n$ matrices $\{(x_{ij})/x_{ij} \in k,\ 1 \leqslant i \leqslant d,\ 1 \leqslant j \leqslant n\}$. For $(a) \in S$, let $D_{(a)}$, $C_{(a)}$ and $C'_{(a)}$ denote the subvarieties of the affine space V^d defined as follows:

$$D_{(a)} = \{(x_{ij}) \in V^d \mid x_{ij} = 0 \text{ for } j < a_i,\ 1 \leqslant i \leqslant d,\ 1 \leqslant j \leqslant n\}$$
$$C_{(a)} = \{(x_{ij}) \in D_{(a)} \mid x_{ia_i} = 1,\ 1 \leqslant i \leqslant d\}$$
$$C'_{(a)} = \{(x_{ij}) \in C_{(a)} \mid x_{ia_k} = 0 \text{ for } i < k,\ 1 \leqslant i, k \leqslant d\}.$$

(Note that $D_{(a)}$, $C_{(a)}$ and $C'_{(a)}$ are affine spaces). Let $\pi : V^d \to \wedge^d(V)$ be the canonical morphism defined by $(v_1, \ldots, v_d) \mapsto b_1 \wedge \ldots \wedge v_d$, $v_i \in V$. It is easy to see that $\pi(D_{(a)}) = \hat{\Omega}_{(a)}$, $\pi(C_{(a)}) = \pi(C'_{(a)}) = U_{(a)}$, and that $\pi : C'_{(a)} \overset{\sim}{\to} U_{(a)}$ is an isomorphism.

REMARK 1.1. When we want to study the properties of a given Schubert variety $\Omega_{(a)}$ or of those contained therein, we can assume $a_1 = 1$ and $a_d < n$.

For, suppose $a_i = a + b_i$, $a \geqslant 0$, $1 \leqslant i \leqslant d$. Then it is easy to see that $\Omega_{(a)}$ is canonically isomorphic to the Schubert variety $\Omega_{(b)}$ in the Grassmannian $G_{d,\, n-a}$ associated to a vector space W of dimension $n - a$. In fact, we can choose W to be the subspace of V spanned by $\{e_i\}$, $a + 1 \leqslant i \leqslant n$. Thus we can assume $a_1 = 1$. Similarly if $a_d = n$, we can identify $\Omega_{(a)}$ with $\Omega_{(a_1, \ldots, a_r)}$ in $G_{r,\, n-d+r}$ for a suitable r such that $r < d$ and $a_r < n - d + r$. In this case W can be choosen to be an $(n - d + r)$-dimensional quotient space of V.

COROLLARY 1.2. *The Schubert varieties of dimension $\leqslant 2$ are isomorphic to the projectivespaces $\mathbf{P}^r_k$, $r \leqslant 2$.*

For, if $\Omega_{(a)}$ is of dimension $\leqslant 2$, it follows from the dimension formula (cf. [**10**], Prop. 5.2, p. 149) that (a) is equal to one of the following elements of S, namely, (max), $(n - d, n - d + 2, \ldots, n)$, $(n - d - 1, n - d + 2, \ldots, n)$ or $(n - d, n - d + 1, \ldots, n)$ assertion follows in view of the above remark.

2. The homogeneous coordinate rings. Let X_{ij} denote the coordinate functions on the affine space V^d. For $(a) \in S$, the coordinate ring of the affine space $D_{(a)}$ is the polynomial ring $K[X_{ij}]_{(a)}$ with $X_{ij} = 0$ for $j < a_i$, $1 \leqslant i \leqslant d$, $1 \leqslant j \leqslant n$. Let $R_{(a)}$ denote the homogeneous coordinate ring of $\Omega_{(a)}$. Recall that $R_{(a)}$ is identified with the subring of $k[X_{ij}]_{(a)}$, generated by the $\{p_{(i)}\}$, $(i) \in S$, where

$$p_{(i)} = \det \begin{vmatrix} X_{1i_1} & \dots & X_{1i_d} \\ \dots & \dots & \dots \\ X_{di_1} & \dots & X_{di_d} \end{vmatrix}$$

Note that $p_{(i)} \neq 0$ if and only if $(a) \leqslant (i)$.

REMARK 2.1. Every non-zero $p_{(i)}$ in $R_{(a)}$ is an irreducible element, i.e., the ideal $(p_{(i)})$ is maximal among the principal ideals in $R_{(a)}$. (This is clear because $R_{(a)}$ is a graded domain generated by the $p_{(i)}$'s, but the $p_{(i)}$'s are of least degree.)

We write $R = R_{(min)}$ for the homogeneous coordinate ring of $G_{d,n}$. Recall that the ideal defining $\Omega_{(a)}$ in $G_{d,n}$ is the ideal $I(S_{(a)})$ in R generated by the $\{p_{(i)}\}$, $(i) \in S_{(a)}$ where $S_{(a)} = \{(i) \in S / i_t < a_t$ for some $t \leqslant d\}$. That is to say $R_{(a)} \approx R/I(S_{(a)})$ (cf. [**10**], 4, p. 154).

3. Opposite cellular decomposition of $G_{d,n}$. The cellular decomposition of $G_{d,n}$ obtained by taking the orbits under the Borel subgroup of $SL(n, k)$ consisting of the *lower* triangular matrices is called the *opposite cellular decomposition* of $G_{d,n}$ (opposite to the one we have fixed). Let —denote the cells in this decomposition. For all (a), $(b) \in S$, we have $\Omega'_{(a)} \subseteq \Omega'_{(b)}$ if and only if $(a) \leqslant (b)$. Consequently, $\Omega'_{(\max)} = G_{d,n}$ and the section of $G_{d,n}$ with the hyper-plane $H_{(\max)}$, whose equation is $p_{(\max)} = 0$, is the codimension 1 Schubert variety in this decomposition. For our future use, we prove the following

PROPOSITION 3.1. *The set-theoretic intersection of any Schubert variety $\Omega_{(a)}$ with the codimension 1 Schubert variety in the opposite cellular decomposition is actually scheme-theoretic and is reduced and irreducible, i.e., $\Omega_{(a)} \cdot H_{(max)}$ is integral (even at the vertex of $\hat{\Omega}_{(a)}$).*

PROOF. We are required to prove that the element $p_{(max)}$ is prime in $R_{(a)}$. To see this, take the polynomial ring $k\,[\bar{X}_{ij}]_{(a)}$ where (the non-zero $\bar{X}_{ij}$'s being indeterminates)

$$\bar{X}_{ij} = 0 \text{ for } \begin{cases} j < a_i,\ 1 \leqslant i \leqslant d \\ j \geqslant n - d + 1,\ i = 1 \\ j > n - d + i,\ 2 \leqslant i \leqslant d. \end{cases}$$

Let $\bar{R}_{(a)}$ denote the subring of this polynomial ring generated by the $d \times d$ minors $\bar{p}_{(i)}$ of the matrix $(\bar{X}_{ij})$. Note that $\bar{p}_{(i)} = 0$ if and only $(i) \in S_{(a)}$ or $(i) = (\max)$. It is clear that we have a natural surjective homomorphism of rings f: $R_{(a)} \to \bar{R}_{(a)}$ defined by $p_{(i)} \longmapsto \bar{p}_{(i)}$. Now proceeding exactly as in the proof of the basis theorem for $\Omega_{(a)}$ (cf. [**10**], Theorem 4.1, p. 155), it is easy to prove that the images under f of the standard monomials in $R_{(a)}$ whose last factors are $\neq p_{(\max)}$ form a vector space basis for $\bar{R}_{(a)}$. Consequently, we get that the kernel of f is precisely the principal ideal $(p_{(\max)})$, hence the result.

REMARKS 3.2 (i) The affine open set $p_{(\max)} \neq 0$ in $G_{d,n}$ is simply the big open cell in the opposite cellular decomposition of $G_{d,n}$ and its coordinate ring, namely $R/(1 - p_{(\max)})R$, can be canonically identified with the polynomial ring $k\,[X_{ij}]$ with the specialisation that

$$X_{i\,n-d+t} = \delta_{it},\ 1 \leqslant i, t \leqslant d.$$

(ii) We shall see (cf. 7, below) that the open set $p_{(max)} \neq 0$ in $\Omega_{(a)}$ is a determinantal locus and conversely.

(iii) One can study the geometry of the scheme-theoretic intersection of a Schubert variety in one cellular decomposition of $G_{d,n}$ with another in the opposite one. Such a study has been made by Chevalley (in [**1**]) for an arbitrary G/B. Some of the results proved there admit easy proofs in the case of $G_{d,n}$.

4. Arithmetic Cohen-Macaulay Character. The following theorem has been proved by several authors, Viz., Hochster, Kempf, Laksov etc. We have given two proofs in [**10**]. Here is another in which we exhibit a canonical system of parameters making a repeated use of Pieri's formula (cf. [**10**], p. 159).

THEOREM 4.1. *Schubert varities are arithmetically Cohen-Macaulay.*

PROOF. Let $\Omega = \Omega_{(a)}$ be a Schubert variety of dimension q. In view of the following lemma, it suffices to prove that $\hat{\Omega}$ is Cohen-Macaulay at its vertex.

LEMMA 4.2. *Let* $W \subseteq \mathbf{P}$ *be a closed subscheme. Then* $\widehat{W}$ *is Cohen-Macaulay if and only if* $\widehat{W}$ *is Cohen-Macaulay at its vertex.*

PROOF. Suppose (0) is a Cohen-Macaulay point of $\widehat{W}$. By [3], 12.1.1, p. 174, $\widehat{W}$ is Cohen-Macaulay in a neighbourhood, say U_0 of (0). It is clear that each fibre of the natural projection $p: \widehat{W} - \{(0)\} \to W$ meets U_0. But these fibres are simply the $\mathbf{G}_m$ — orbits in $\widehat{W} - \{(0)\}$, and hence $\widehat{W}$ is Cohen-Macaulay at each point of $p^{-1}(x)$ for all $x \in W$. This proves the Lemma.

Proof of the theorem (continued). Let S_j, $j = 0, \ldots, q$, denote the subsets of S defined as follows:

$$S_j = \{(b) \in S/(a) \leqslant (b) \text{ and } \sum_{i=1}^{d}(b_i - a_i) = j\}.$$

Clearly $S_0 = \{(a)\}$. Since $q = \Sigma(n - a_i) - \frac{1}{2} d(d-1)$ (by the dimension formula), it follows that $S_{q-1} = \{(n-d,\ n-d+2,\ \ldots n)\}$ and $S_q = \{\max)\}$. Let $f_j = \sum_{(t) \in S_j} p_{(t)}.\ j = 0, \ldots, q$.

Claim: $f_0, \ldots, f_q$, *is the required system of parameters for* $\hat{\Omega}$ *at its vertex.*

To see this, first note the following. Let $\Omega' \subset \Omega$ be a Schubert variety of dimension r. Construct $f_0', \ldots, f_r'$ as above for Ω'. It is easy to see that $f_0, \ldots, f_{q-r-1} \equiv 0 \bmod I(\Omega')$ and $f_{q-r+j} \equiv f_j' \bmod I(\Omega')$ for $j = 0, \ldots, r$, where $I(\Omega')$ is the ideal defining Ω' in Ω.

Now the rest of the proof is the same as our second proof in [10], p. 170.

5. Arithmetic Normality.

THEOREM 5.1. *Schubert varieties are arithmetically normal.*

PROOF. Let $\Omega = \Omega_{(a)}$ be a Schubert variety. Since $\hat{\Omega}$ is Cohen-Macaulay, it suffices to prove that the singular locus, say Z_0 of Ω is of codimension $\geqslant 2$ in Ω. We know that Z_0 is a union of some of the Schubert varieties contained in Ω, and so it suffices to show that none of the codimension 1 Schubert varieties, say $\Omega_1, \ldots, \Omega_p$ in Ω is contained in Z_0. Let I_i be the ideal defining Ω_i in Ω, $1 \leqslant i \leqslant p$. Recall that, by Pieri's formula, we have in the ring $R_{(a)}$

$$(p_{(a)}) = \bigcap_{i=1}^{p} I_i$$

This shows that, if x_i is the generic point of Ω_i, the local ring $\mathcal{O}_{\Omega, x_i}$ is a discrete valuation ring (whose maximal ideal is generated by $p_{(a)}$), and hence $x_i \not\subset Z_0$ for any i. Hence the result.

6. Picard group of Schubert Varieties.

THEOREM 6.1. *The Picard group of a non-trivial Schubert variety is* $\mathbf{Z}$ *and is generated by the class of* L.

PROOF. Let $\Omega = \Omega_{(a)}$ be a Schubert variety of dimension $q \geqslant 1$. We prove the result by induction on q. For $q \leqslant 2$, by Corollary 1.2 above, Ω is a projective line or plane and so the result follows. Assume that $q \geqslant 3$.

Let H be the hyper-plane in $\mathbf{P}$ whose equation is $P_{(a)} = 0$. By Pieri's formula, we know that $Z = \Omega . H$ is a union of the Schubert varieties Ω_i, $1 \leqslant i \leqslant p$ of codimension 1 in Ω. Hence by the induction hypothesis, the theorem is true for the components of Z. Now we make the

CLAIM A: *Pic* $Z = \mathbf{Z}$ *and is generated by the class of* L.

To see this, we proceed by induction on p, the number of components of Z. We can assume $p \geqslant 2$. As in [**10**] p. 164, write

$$X = \bigcup_{i=1}^{p-1} \Omega_i, \; Y = \Omega_p \text{ and } Z' = X \cap Y.$$

By induction on p and q, it follows that the Picard groups of X, Y and Z' are Z and are generated by L. Note that dim $Z' = q - 2 \geqslant 1$). Now look at the sequence of groups:

$$(*) \qquad 0 \longrightarrow \text{Pic } Z \xrightarrow{\varphi} \text{Pic} \times \text{Pic } Y \xrightarrow{\psi} \text{Pic } Z' \longrightarrow 0$$

where φ is the restriction map and ψ takes (L^r, L^s) to L^{r+s}. Now Claim *A* is a consequence of the

CLAIM B: *The sequence* (*) *is exact.*

Clearly ψ is surjective and $\operatorname{Im}\varphi = \operatorname{Ker}\psi$. To see φ is injective: recall that we have an exact sequence of $\mathcal{O}_Z$-modules (cf. [**10**], p. 165)

(**) $$0 \to \mathcal{O}_Z \to \mathcal{O}_X \oplus \mathcal{O}_Y \to \mathcal{O}_{Z'} \to 0$$

Let M be a line bundle on Z whose class is in Ker φ. Now (**) induces an exact sequence

$$0 \to M \to M|_X \oplus M|_Y \to M|_{Z'} \to 0$$

i.e., the sequence

$$0 \to M \to \mathcal{O}_X \oplus \mathcal{O}_Y \to \mathcal{O}_{Z'} \to 0$$

is exact. Since X, Y and Z' are *connected* projective schemes, we get an exact sequence of vector spaces

$$0 \to H^\circ(Z, M) \to k \oplus k \to k \to 0$$

This gives that $H^\circ(Z, M) \approx k$. Similarly, replacing M by M^{-1}, we get that $H^\circ(Z, M^{-1}) \approx k$. But this is possible only if $M \approx \mathcal{O}_Z$. Hence Claim *B* is proved.

Now the theorem follows from Claim *A* and the final

CLAIM C: *The natural restriction map Pic* $\Omega \to$ *Pic Z is injective (and hence an isomorphism).*

To see this, we need the following

THEOREM 6.2. (*Grothendieck cf.* [**2**], *Cor.* 3.6, *Exp. XII, p.* 153)). *Let T be a projective scheme (over k) of dimension* $\geqslant 3$. *Let* $\mathcal{O}_T(1)$ *be an ample line bundle on T, and let t be a section of* $\mathcal{O}_T(1)$ *which is* $\mathcal{O}_T$*-regular. Let* T_0 *be the divisor of the zeros of t. Assume that T has depth* $\geqslant 2$ *at each of its closed points and that* $H^1(T_0, \mathcal{O}_{T_0}(r)) = 0$ *for* $r < 0$. *Then for every open neighbourhood U of* T_0, *the natural map Pic* $U \to$ *Pic* T_0 *is injective (in particular, Pic* $T \to$ *Pic* T_0 *is injective).*

The hypothesis of the theorem is satisfied for $T = \Omega$, $\mathcal{O}_T(1) = L$, $t = p_{(a)}$ and $T_0 = Z$ (in view of Theorem 4.1 above, and the vanishing theorems for Z (cf. [**10**], Step A. p. 164)). This proves Claim *C* and hence also the theorem.

7. Determinantal Loci. Definition 7.1. Let $1 \leqslant k \leqslant d \leqslant n$ be fixed integers. Let $0 \leqslant a_1 < \ldots < a_k = n$ be a sequence of integers. Let $\{U_{ij}\}$, $1 \leqslant i \leqslant d$, $1 \leqslant j \leqslant n$, be a $d \times n$ matrix of indeterminates over the field k, and let $k[U_{ij}]$ be the polynomial ring. Let $J_{a_1 \cdots a_k}(d, n)$ denote the ideal in $k[U_{ij}]$ generated by the $t \times t$ minors of the $d \times a_t$ submatrix (U_{ij}), $1 \leqslant i \leqslant d$, $1 \leqslant j \leqslant a_t$ for all $t = 1, \ldots, k$. Then the subscheme of the affine space Spec $(k[U_{ij}])$ defined by the ideal $J_{a_1 \cdots a_k}(d, n)$ is called a *determinantal locus*, and is denoted by $D_{a_1 \cdots a_k}(d, n)$. Since no confusion is likely, we write $J_k(d, n)$ for the case when $a_t = t - 1$, $1 \leqslant t \leqslant k - 1$, $a_k = n$. The corresponding determinantal locus, denoted by $D_k(d, n)$, is simply the locus of $k \times k$ minors of the matrix (U_{ij}).

Denote by $D^+_{a_1 \cdots a_k}(d, n)$ the subscheme of the projective space Proj $(k[U_{ij}])$ defined by the (homogeneous) ideal $J_{a_1 \cdots a_k}(d, n)$. We call $D^+_{a_1 \cdots a_k}(d, n)$ the *projective determinantal locus*. By definition, $D_{a_1 \cdots a_k}(d, n)$ is the cone over $D^+_{a_1 \cdots a_k}(d, n)$. We will continue to denote the vertex of $D_{a_1 \cdots a_k}(d, n)$ by (0).

REMARK. 7.2. The projective determinantal locus $D^+_2(d, n)$ is non-singular (and hence the vertex of $D_2(d, n)$ is an isolated singularity unless $d = 1$).

To see this, consider the Segre imbedding

$$s: \mathbf{P}_k^{d-1} \times \mathbf{P}_k^{n-1} \to \mathbf{P}_k^{dn-1}$$

Recall that the image of s is an intersection of quadrics in $\mathbf{P}_k^{dn-1} =$ Proj $(k[U_{ij}])$, and the equations of these quadrics are nothing but the 2×2 minors of the matrix (U_{ij}). In other words, the image of s is simply $D^+_2(d, n)$ and hence the assertion.

The following theorem establishes the connection between determinantal loci and Schubert varieties. Our treatment is based on an observation due to Hochster (cf. [4], Corollary (3.13), p. 53). We bring out the underlying idea therein. Interpreted properly, we find that the determinantal loci are nothing but the scheme-theoretic intersection of Schubert varieties with the big open cell in the opposite cellular decomposition of the Grassmann variety. Kleiman (cf. [7] 4.8, p. 424) has done this for the class of the determinantal loci $D_k(d, n)$.

THEOREM 7.3. *The determinantal locus* $D_{a_1\ldots a_k}(d,n)$ *is canonically isomorphic to the standard affine open subset* $p_{(n+1,\ldots,n+d)}=p_{(\max)}\neq 0$ *of the Schubert variety* $\Omega_{(b)}$ *in* $G_{d,n+d}$ *where* $(b)=(b_1,\ldots,b_d)$ *is defined by* $b_i=a_i+1$ *for* $1\leqslant i\leqslant k$ *and* $b_j=n+j-k+1$ *for* $k\leqslant j\leqslant d$. *Consequently,* $D_{a_1\cdots a_k}(d,n)$ *is*

...integral (i.e., the ideal $J_{a_1\cdots a_k}(d,n)$ *is prime)*

...Cohen-Macaulay (i.e., $J_{a_1\cdots a_k}(d,n)$ *is "perfect")*

...normal

...of dimension $=\dim\Omega_{(b)}$

$$=\tfrac{1}{2}(k-1)(2n+2d-k)-\sum_{i=1}^{k-1}a_i$$

PROOF. For convenience, write $J=J_{a_1\cdots a_k}(d,n)$, $D=D_{a_1\cdots a_k}(d,n)$ and $\Omega=\Omega_{(b)}$ with (b) as in the Theorem. Write $I=I(S_{(b)})$, the ideal of Ω and $A=k[U_{ij}]/J$, the coordinate ring of D. Let $B=R/I$ be the coordinate ring of Ω where R is the coordinate ring of $G_{d\ n+d}$. Recall that R is identified with the subring of $k[X_{ij}]$, $1\leqslant i\leqslant d$, $1\leqslant j\leqslant n+d$, generated by the $d\times d$ minors of the matrix (X_{ij}).

We are to prove that A is isomorphic to $B'=B[p_{(\max)}^{-1}]_0$, the coordinate ring of the affine open subset $p_{(\max)}\neq 0$ of Ω. We have

$$\begin{aligned} B' &= B/(1-p_{(\max)})B \\ &= R/(I+(1-p_{(\max)})R) \\ &= R/(1-p_{(\max)})R/(I+(1-p_{\max})R)/(1-p_{(\max)})R \end{aligned}$$

Now look at $R/(1-p_{(\max)})R=R[p_{(\max)}^{-1}]_0$ which is the coordinate ring of the big open cell in the opposite cellular decomposition of $G_{d,n+d}$. By Remark 3.2 (i) above, we know that $R/(1-p_{(\max)})R$ is canonically identified with the polynomial ring $k[X_{ij}]$, $1\leqslant i,\leqslant d, 1\leqslant j\leqslant n$ (i.e. with the specialisation $X_{i,\,n+t}=\delta_{it}$, $1\leqslant i,t\leqslant d$). Under this identification, it is clear that $p_{(\max)}=1$, $R/(1-p_{(\max)})R\approx k[U_{ij}]$ and I corresponds precisely to the ideal J. Thus $B'\approx A$ as required. The other assertions are immediate since D is open in Ω and Ω has those properties.

REMARKS. (1) It is easy to see that the vertex of D corresponds to the point x_0 of Ω where $\{x_0\}=\Omega_{(\max)}$ is the point Schubert variety.

(2) While the determinantal locus D inherits all the "good" properties of Ω, it is *not* clear *a priori* if Ω reflects the good properties of D (except normality and non-singularity). However, we will see that the property being factorial is well behaved (even for the cone $\hat{\Omega}$) (cf. Theorem 8.1, below).

(3) Suppose Ω is a Schubert variety in $G_{r,s}$, and D is the affine open subset $p_{(\max)} \neq 0$ in Ω, then D *does not* really correspond to a determinantal locus as defined above. However, one can generalise the definition of a determinantal locus so as to include these cases. In other words, D is a "true" determinantal locus. But what is useful is to note that this affine open subset is a cone (i.e., its coordinate ring is graded) and the above remarks apply. In other words, Schubert varieties are cones in a neighbourhood of x_0 (with x_0 being the vertex). (We shall use only this fact in the sequel).

It is easy to see that the first q elements of the canonical system of parameters $f_0, \ldots, f_q$ ($q = \dim \Omega$) at the vertex of $\hat{\Omega}$ (constructed in §4, above) go down to a system of parameters at the vertex of D.

(4) Determinantal loci, being cones and at the some time open subvarieties in Schubert varieties in a particular way, admit a good algebro-geometric study as for the Schubert varieties.

(5) A determinantal locus of dimension $\leqslant 2$ is an affine space. (This is so because the corresponding Schubert varieties are linear (Corollary 1.2, above)).

8. Characterisation of factorial Schubert varieties.

THEOREM 8.1. *Let $\Omega_{(a)}$ be a Schubert variety in $G_{d,n}$ and let $D_{(a)}$ be the determinantal locus (i.e., the affine open subset $p_{(\max)} \neq 0$) in $\Omega_{(a)}$. Then the following statements are equivalent:*

(*i*) *$\Omega_{(a)}$ is arithmetically factorial (i.e., the ring $R_{(a)}$ is a UFD).*

(*ii*) *There exists a unique Schubert variety of codimension 1 in $\Omega_{(a)}$.*

(*iii*) *The element $p_{(a)}$ is prime in $R_{(a)}$.*

(*iv*) *$(a) = (a_1, \ldots, a_d)$ is constituted by either one or at most two segments of successive integers according as $a_d \leqslant n$.*

(v)* $\Omega_{(a)}$ *is isomorphic to the Grassmann variety* $G_{r,s}$ *for a suitable r and s.*

(vi) $\Omega_{(a)}$ *is non-singular*

(vii) $\Omega_{(a)}$ *is (locally) factorial*

(viii) $D_{(a)}$ *is factorial (at its vertex).*

PROOF. Let $\Omega_1 \ldots, \Omega_p$ be all the Schubert varieties of codimension 1 in $\Omega_{(a)}$. Let I_i be the ideals defining the Ω_i 's in $\Omega_{(a)}$. Recall that the ideals I_i are of height 1 in $R_{(a)}$ and that the element $p_{(a)}$ is irreducible (cf. Remark 2.1 above). Further, by Pieri's formula we have

$$(p_{(a)}) = \bigcap_{i=1}^{p} I_i$$

With these observations, it is trivial to see that (i) ⇒ (ii) ⇒ (iii).

Now assume (iii) is true. If (iv) were not true, there would exist (b), $(c) \in S$, such that $(b) \neq (c)$, $(a) \leqslant (b)$, $(a) \leqslant (c)$ and

$$\sum_{i=1}^{d} (b_i - a_i) = 1 = \sum_{i=1}^{d} (c_i - a_i)$$

This shows that the ideals $I(S_{(b)})$ and $I(S_{(c)})$ (corresponding to the Schubert varieties $\Omega_{(b)}$ and $\Omega_{(c)}$ of codimension 1 $\Omega_{(a)}$) are distinct prime ideals of height 1 in $R_{(a)}$ and contain the prime ideal $(p_{(a)})$. This is clearly a contradiction. Hence (iv) holds.

By Remarks 1.1, (iv) ⇒ (v) is immediate. It is obvious that (v) ⇒ (vi) ⇒ (vii) ⇒ (viii). Finally, (viii) ⇒ (i):

First look at the following.

LEMMA 8.2. *(cf. Samuel* **[12]**, *Prop.* 7.4, *p.* 26). *Let* $X \subset \mathbf{P}$ *be a closed subscheme and A its homogeneous coordinate ring (so that* $\hat{X} = \operatorname{Spec} A$). *(Assume that* $\hat{X}$ *is normal). Then* $\hat{X}$ *is factorial at its vertex if and only if A is a UFD.*

Since $D_{(a)}$ is normal and a cone (cf. Theorem 7.3 and Remark 7.4 (3) above), by hypothesis and the above lemma, we get that the coordinate ring $R'_{(a)} = R_{(a)} [p^{-1}_{(\max)}]_0$ of $D_{(a)}$ is a *UFD*. But then $R_{(a)} [p^{-1}_{(\max)}] = R'_{(a)} [p_{(\max)}, p^{-1}_{(\max)}]$ is a *UFD* because $p_{(\max)}$ is transcendental over $R'_{(a)}$. By Pro-

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*I am grateful to R.C. Cowsik who has drawn my attention to this position 3.1 above, we know that $p_{(\max)}$ is a prime element in $R_{(a)}$. Hence it follows that $R_{(a)}$ is a *UFD*. Thus (i) holds. This completes the proof of the Theorem.

From the statement (iv) of the above Theorem, it is easy to see the following.

COROLLARY 8.3. (1) *The number of non-singular Schubert varieties in (a fixed cellular decomposition of) $G_{d,n}$ is equal to $1 + \dim G_{d,n} = 1 + d(n-d)$.*

(2) *The codimension of a non-singular Schubert variety ($\neq G_{d,n}$) is at least equal to min $(d, n-d)$.*

(3) *$G_{d,n}$ is isomorphic to the projective space $\mathbf{P}_k^{d(n-d)}$*
$\Leftrightarrow$ the non-singular Schubert varieties in $G_{d,n}$ form a chain
$\Leftrightarrow$ the Schubert variety of codimension 1 in $G_{d,n}$ is non singular
$\Leftrightarrow d = 1$ or $n-1$.

COROLLARY 8.4. *The vertex of a determinantal locus D is factorial if and only if D is isomorphic to an affine space.*

PROOF. If the vertex of D is factorial, D is already non-singular because D is open in a Grassmann variety (by the above theorem). But then D, being a cone, is non-singular at its vertex implies it is an affine space.

PARAFACTORIALITY. For the definition of a parafactorial couple (X, Y), (see [2] Exp. *XI*, p. 126 or [3], 21.13, p. 313). Note that for a *normal* projective variety $X \subset \mathbf{P}$, if X is arithmetically normal (i.e., $\text{depth}_{(0)} \hat{X} \geqslant 2$), Pic $X = \mathbf{Z}$ (and generated by the class of $\mathcal{O}_X(1)$) is equivalent to saying that $(\hat{X}, (0))$ is a parafactorial couple. Thus for a non-trivial Schubert variety Ω by Theorems 5.1 and 6.1 above, we find that $\hat{\Omega}$, is parafactorial at its vertex.

If D is a determinantal locus of dimension $\geqslant 2$, it is natural to ask if D is parafactorial at its vertex. The answer is *no* because D is normal but in general Pic $D^+ \neq \mathbf{Z}$, for example Pic $D_2^+(d,n) = \mathbf{Z} \oplus \mathbf{Z}$ (cf. Remark 7.2 above).

If the vertex (0) of D is a singularity of D, then (0) remains non-factorial by Corollary 8.4, above. Now we have the following.

Corollary 8.5. *If the vertex* (0) *of a determinantal locus D is the only non-factorial point of D, then D is not parafactorial at* (0) (*i.e.* $Pic\, D^{+} \neq \mathbf{Z}$).

PROOF. By Remark 7.4 (5) above, and hypothesis, we get that dim $D \geqslant 3$. But then if (D, (0)) were a parafactorial couple, by [2], Cor. 3.10, p. 130, we would get that (0) is a factorial point of D which is a contradiction.

Errata to "Postulation formula for Schubert varieties"

(See [**10**] under the references).

On page 161, the statement of Proposition 1.2 should read as "... the local rings $\mathcal{O}_{\hat{Y}_{i},(0)}$ are Cohen-Macaulay and have dimensions $= d_i$, ..." and in its proof, (i) should read as "Suppose R is a Cohen-Macaulay noetherian ...". On page 168, lines 2, 3 and 4 from top must be modified as follows: "By assertions (1) and (2) of the theorem for X, Y and Z', and Proposition 1.1, we get that the local rings $R_{X,(0)}$, $R_{Y,(0)}$ and $R_{Z',(0)}$ are Cohen-Macaulay. Now by ...".

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