Journal of Mines, Metals and Fuels, *71(11): 2327-2333; 2023. DOI: 10.18311/jmmf/2023/36259*

Print ISSN : 0022-2755

Journal of Mines, Metals and Fuels

Contents available at: *www.informaticsjournals.com/index.php/jmmf*

Study on Nanofluid Boundary Layer Flow Over a Stretching Surface by Spectral Collocation Method

M. S. Gayathri[1]*, N. P. Bhavya[2], P. A. Dinesh[3] and Chandrashekar Badachi[4]

[1]*Department of Mathematics, B.M.S College of Engineering, Bangalore - 560019, Karnataka, India; msgayathri.maths@bmsce.ac.in,*
[2]*Department of Mathematics, Sapthagiri College of Engineering, Bangalore – 560057, Karnataka, India; bhavya201983@gmail.com*
[3]*Department of Mathematics, Ramaiah Institute of Technology, Bangalore - 560054, Karnataka, India; dineshdpa@msrit.edu*
[4]*Department of Electrical and Electronics Engineering, Ramaiah Institute of Technology, Bangalore – 560054, Karnataka, India; badachi@msrit.edu*

Abstract

The method of Spectral collocation is used to analyze the flowing Nano fluid layer in contact with a stretching surface for comprehensive information and thus to have its utility in industrial activities like the production of glass fibers, petroleum refining, hot rolling of metals, metal spinning etc. The spectral collocation model incorporates thermophoresis and Brownian motion phenomena to describe the fluid flow, fluid concentration and temperature profiles. A similarity solution has been presented for the governing equations of fluid momentum, concentration and temperature. The computational results are the function of Prandtl number (Pr), Lewis number (Le), thermophoresis and Brownian motion phenomena. The engineering quantities like thermophoresis parameter (Nt), Brownian motion parameter (Nb), buoyancy-ratio parameter (Nr) and reduced Nusselt number (Nu) and reduced Sherwood number (Sh) have tabulated corresponding to Prandtl number (Pr) and Lewis number (Le). The results of the current study thrown light on fluid velocity and heat transfer rates in the boundary layer. The numerous industrial products and manufacturing processes of superior quality can be exercised with the current studies.

Keywords: *Brownian Motion, Fluid Velocity, Heat Transfer Rate, Nano Fluid, Thermophoresis*

1.0 Introduction

A key aspect in fluid dynamics is the flow across a stretching surface. Several engineering techniques used in glass-making, wire drawing, melt-spinning, hot rolling and extrusion, glass fiber and rubber sheets synthesis, polymer processing, petroleum refining depends on the behavior of the fluid flow across a stretching surface. The quality of the finished goods is significantly impacted by both the stretching kinematics and the concurrent heating or cooling during such procedures. Takhar *et al.*, has investigated unsteady laminar flow of fluid over solid surface with the use of implicit finite difference method[1]. The flow behavior comparison at constant velocity on continuous surfaces and on surface boundary layers of fixed length has also carried out[2]. Crane comprehensively studied on fluid flow across a stretching plane which was moving in its plane with linear velocity due to uniform stress[3]. The phenomena have well described with the help of 2-D Navier–Stokes equations. The solution of the stretching sheet with variation in temperature and the magnetic field makes boundary layer thin and increases

**Author for correspondence*

wall friction[4-6]. The topic of convective transport of nano fluids has been attracting researchers globally. The heat transfer rate control of nano fluids with suspensions of nano solid particles which are metallic or non-metallic been studied and the resulting nano fluids exhibiting good thermal conductivities[7-8].

According to the research by Choi *et al*.[9], the fluid's thermal conductivity enhances by about two times on adding less than one percent of nanoparticles by volume in liquids of conventional heat transfer. The solid particle dispersion and the efficiency of nanofluids in heat transmission were studied[10]. Many researchers felt that nanotechnology will play a significant role lead to industrial revolution. It tries to manipulate matter's molecular structure with the intention of bringing about innovation in almost every sector of the economy and in public projects such as national security, physical and medical sciences, electronics cooling and transportation.

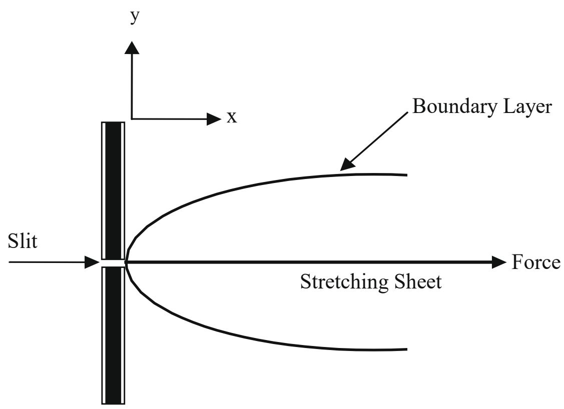
Thermal conductivity and convective heat flow are two topics covered in some computational and experimental investigations on nano fluids[11-14]. An anomalous convective situation seen in nano fluids. It was reported that reduced Nusselt number is a function of nano fluid buoyancy-ratio parameter (N_r), Brownian motion parameter (N_b) and thermophoresis parameter (N_t). Further stated that thermophoresis and Brownian diffusion found dominant in the vicinity of turbulent effects[15-17]. Cheng and Minkowycz have studied natural steady convection onto a vertical plane immersed in a saturated porous medium of high Rayleigh numbers[18]. W. A. Khan and I. Pop have studied the phenomena of fluid laminar flow across the stretching sheet using numerical methods[19].

As a corollary to finite difference methods, partial differential methods can also be employed. The spectral methods which use partial differential methods are global methods, in which the calculation at any given position depends on data from the whole area. The spectral methods been more precise than local approaches. Nischay Raj and Sabyasachi Mondal have done an extensive review on spectral methods for various fluid flow types particularly the nano fluid flow[22]. Ghasemi *et al.,* have noticed the heat transfer of nano fluid nonlinear flow over a stretching surface in the presence of magnetic field using spectral relaxation[20]. Samuel and Motsa have formulated a tri-variate spectral collocation method to explain 2-D non-linear partial differential expressions of some regular geometry[21]. Seyed Mahdi Mousavi *et al*., have studied 2-D magneto hydro dynamics of a nanofluid with base fluid as water and Ag, MgO as the nanoparticles moving over a stretching sheet with suction, radiation and convective boundary condition effects using the software MATLAB[23]. A close agreement of numerical results with existing data was noticed. Srinivasulu and Shankar analysed Williamson fluid subjected to magnetic field and depicted the concentration, velocity and temperature variations with convective boundary conditions using R-K Fehlberg's 4^{th}-5^{th} order and shooting technique[24]. In a stretching sheet which is non-linear the boundary layer flow and transfer of heat are studied and the effect of Prandtl number, Brownian motion parameter and thermophoresis parameter are considered by Vishwanath *et al*., and concluded that Nb and Nt increases because of rise in mass transfer rates[25]. Kavya and Nagendramma studied the impact of temperature slip and velocity jump combined with variable thermal effects, Boussinesq approximation and dissipation effects of a hybrid nanofluid based on Newtonian and non-Newtonian principles under mixed convection effects across a stretching cylinder[26]. Syed and Srinivas concluded that adding the aerospike to the hypersonic vehicle produced drag reduction results under particular flow condition[27].

The earlier research studies been planned to complement with additional important information through this current work. In the current study, the spectral collocation approach been exercised to explore the flow of nano fluid across a stretching plane, nano particle fraction and heat transfer phenomena. The governing equations of fluid flow have reduced to a two-point boundary condition problem with similarity parameters and resolved using spectral collocation method. The impact of major variables on fluid velocity, particle concentration and fluid temperature are highlighted through graphical representations.

2.0 Equations of Convective Transport

In current study, a linear stretching surface across which 2-dimensional incompressible nano fluids flow has been considered as presented below. The stretching plane velocity is $u_w(x)$ = ax where 'a' and 'x' are a constant



Problem Geometry

and stretching sheet coordinate respectively. The flow begins at y ≥ 0 where y is the perpendicular coordinate to stretching sheet. The sheet is stretched along x-axis by applying constant uniform stress. The temperature and nanoparticle concentration of the fluid near surface are denoted by T_w and C_w. At large values of 'y', nanoparticle concentration and temperature denoted by C_∞ and T_∞.

The following are the expressions of continuity, momentum, energy and concentration of an incompressible nano fluid flow.

Continuity equation of incompressible nano fluid flow

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=0 \tag{1}$$

Momentum transfer equation of incompressible nano fluid flow

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho_f}\frac{\partial p}{\partial x}+\upsilon\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \tag{2}$$

Energy Transfer equation of incompressible nano fluid flow

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho_f}\frac{\partial p}{\partial y}+\upsilon\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right) \tag{3}$$

Concentration equation of incompressible nano fluid flow

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+$$

$$\tau\left\{D_B\left(\frac{\partial C}{\partial x}\frac{\partial T}{\partial x}+\frac{\partial C}{\partial y}\frac{\partial T}{\partial y}\right)+\left(\frac{D_T}{T_\infty}\right)\left[\left(\frac{\partial T}{\partial x}\right)^2+\left(\frac{\partial T}{\partial y}\right)^2\right]\right\} \tag{4}$$

$$u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}=D_B\left(\frac{\partial^2 C}{\partial x^2}+\frac{\partial^2 C}{\partial y^2}\right)+$$

$$\left(\frac{D_T}{T_\infty}\right)\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right) \tag{5}$$

Subjected to boundary conditions

$$v=0, u=u_w(x)=ax, T=T_w, C=C_w \; at \; y=0$$

$$u=0, v=0, T=T_\infty, C=C_\infty \; as \; y=\infty \tag{6}$$

The horizontal and vertical velocities denoted by u and v, fluid pressure denoted by p, fluid density by ρ_f, kinematic viscosity by υ, fluid thermal diffusivity by α. Let $\tau=(\rho c)_p/(\rho c)_f$ presents proportion of heat capacities with ρ as density, D_B is co-efficient of Brownian diffusion, D_T is co-efficient of thermophoresis diffusion.

The stream function (Ψ) can be written as u = ∂Ψ/∂y and v = -∂Ψ/∂x. Let η represents similarity variable, θ(η) is dimensionless temperature and ϕ(η) is nanoparticle volume fraction. Then the following dimensionless quantities can be introduced now as;

$$\Psi=(av)^{1/2}xf(\eta), \eta=(a|v)^{1/2}y$$

$$\theta(\eta)=\frac{T-T_\infty}{T_w-T_\infty}, \phi(\eta)=\frac{C-C_\infty}{C_w-C_\infty} \tag{7}$$

Substituting equation (7) into equation (2)- (5) the governing equations (1)-(5) takes the form as follows;

$$f'''+ff''-f'^2=0 \tag{8}$$

$$\frac{1}{Pr}\theta''+f\theta'+Nb\phi'\theta'+Nt\theta'^2=0 \tag{9}$$

$$\phi''+Lef\phi'+\frac{Nt}{Nb}\theta''=0 \tag{10}$$

The reduced relevant boundaries are

$$f(0)=0, \theta(0)=1, \phi(0)=1$$

$$f'(0)=1, f'(\infty)=0, \theta(\infty)=0, \phi(\infty)=0 \tag{11}$$

The key variables in equation (8) – (10) are given by Lewis number (Le), buoyancy-ratio parameter (Nb) Prandtl number (Pr), and thermophoresis parameter (Nt) as follows;

$$Pr = \frac{\upsilon}{\alpha}\ ;\ \ Le = \frac{\upsilon}{D_B};\ \ Nb = \frac{(\rho c)_P D_B(\phi_w - \phi_\infty)}{(\rho c)_f \upsilon};$$
$$Nt = \frac{(\rho c)_P D_T(T_w - T_\infty)}{(\rho c)_f T_\infty \upsilon} \tag{12}$$

The analytical solution of equation (8) is given by equation (13);

$$f(\eta) = 1 - e^{-\eta} \tag{13}$$

Let q_m and q_w are local mass flux and local heat flux of the surface. The Nusselt number, Sherwood number and Reynolds number; dimensionless variables can be given by;

$$Nu = \frac{xq_w}{k(T_w - T_\infty)}\ ;\ \ Sh = \frac{xq_m}{D_B(C_w - C_\infty)}\ ;$$
$$Re_x = u_w(x)x/\upsilon \tag{14}$$

Using the non- dimensional variables, we get

The reduced Nusselt number (Nu_r) and reduced Sherwood number (Sh_r) at the surface of the sheet can be written as given in equation (15).

$$Nu_r = {Re_x}^{-1/2} Nu = -\theta'(0)\ ,$$
$$Sh_r = {Re_x}^{-1/2} Sh = -\phi'(0) \tag{15}$$

In current study the differential expressions (8) – (10) subjected to the boundaries cited in equation (11) have been solved by quasi-linear spectral collocation numerical method. The transformation η=L/2 (z+1) converts the physical domain in η ϵ [0, L] to the computational domain z ϵ [-1, 1]. The L is a big enough number to roughly represent the boundary condition at η ⟶ ∞. The Gauss-Lobatto points provided by z_j=cos(Πj/N) j=0(1) N are utilized to solve the linearized equations using the Chebyshev spectral collocation method. The collocation points are denoted by N. In this method of solution, the resulted system of equations from discretization has been solved in MATLAB using pseudo inverse operator.

3.0 Results and Discussion

The governing expressions (8)-(10) subjected to condition (11) were numerically solved using Spectral collocation method. The current outcomes of spectral collocation verified by comparing the same with Khan and Pop[19] from accuracy verification point of view and the same has shown in the table below. The Table presents reduced Nusselt number (Nu_r) and reduced Sherwood number (Sh_r) for fixed *Pr =10, Le=10, Nb=0.1* noticed in the current study. The graphs plotted to depict the impact of variations in Prandtl number (*Pr*), Lewis number (*Le*), Brownian motion (*Nb*) and thermophoresis parameter (*Nt*) on fluid particle concentration profile and temperature profile.

The Figure 1 presents variations in fluid temperature with changes of fluid thermophoresis parameter as 0.1, 0.2, and 0.3. The rise in fluid temperature with the increased *Nt* attributed to decreased conductivity of nanoparticles and increased width of boundary layer led to particles

Table 1. Values of Nu_r and reduced Sh_r for fluid parameter *Nt* with fixed entries of *Pr =10, Le=10, Nb=0.1*

Thermophoresis parameter (Nt)	Reduced Nusselt Number (Nu_r)		Reduced Sherwood Number (Sh_r)	
	As per study of W. A. Khan and I. Pop[19]	As per current study by Spectral collocation method	As per study of W. A. Khan and I. Pop[19]	As per current study by Spectral collocation method
0.1	0.9524	0.9522	2.1294	2.1294
0.2	0.6932	0.6934	2.2740	2.2742
0.3	0.5201	0.5203	2.5286	2.5287
0.4	0.4026	0.4028	2.7952	2.7953
0.5	0.3211	0.3214	3.0351	3.0353

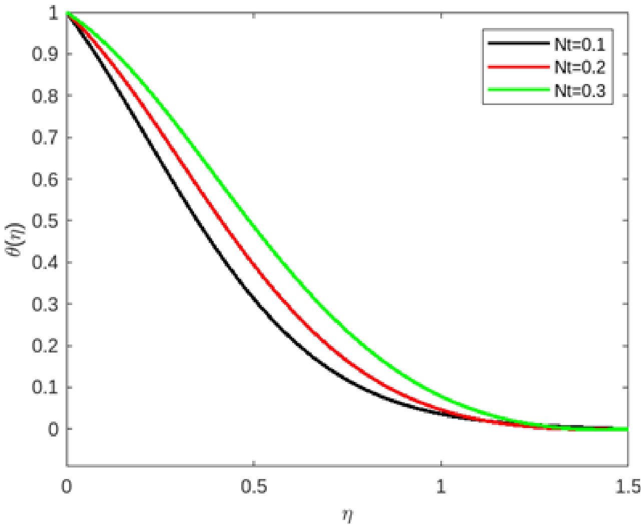


Figure 1. Variations in temperature profile θ(η) with thermophoresis parameter (Nt).

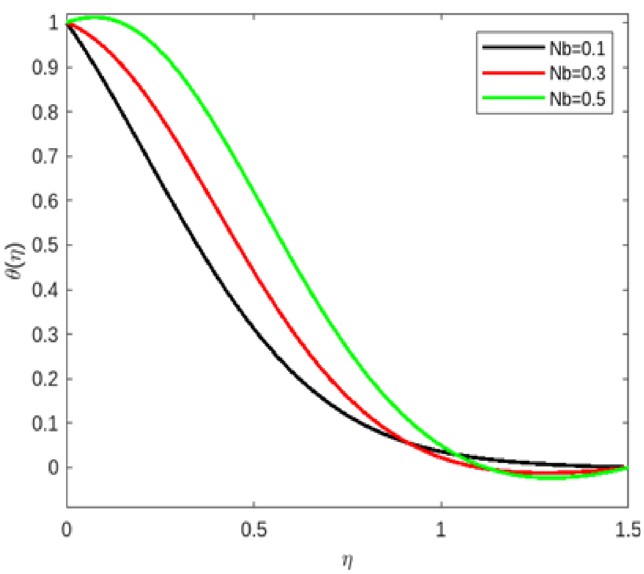


Figure 2. Variations in temperature profile θ(η) with Brownian motion (Nb).

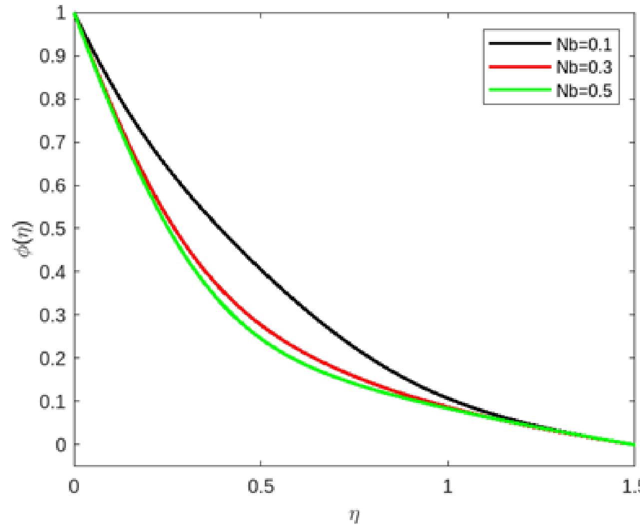


Figure 3. Variations in concentration profile ϕ(η) with Brownian motion (Nb)

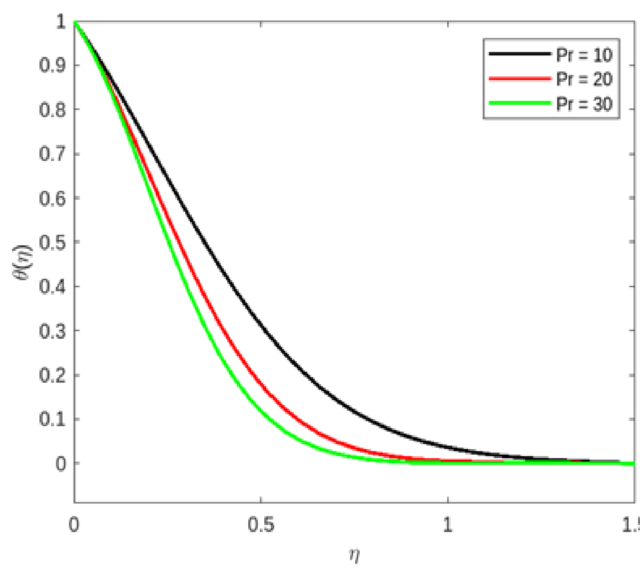


Figure 4. Variations in temperature profile θ(η)with Prandtl number (Pr).

reallocation from hotter to colder zone and hence fluid temperature rises.

The variations in the temperature distribution θ(η) with Brownian motion (Nb) of 0.1, 0.2 and 0.3 is shown in Figure 2. The fluid particles striking creates Brownian motion and thus enhances the width of boundary layer and causes increase in the temperature.

The fluid concentration profile ϕ(η) also influenced by Brownian motion (*Nb*) of 0.1, 0.2 and 0.3 and the same has presented in Figure 3. It is evident that boundary layer thickness and concentration profiles declines with an increase in Brownian motion (*Nb*).

The variation of temperature distribution θ(η) against Prandtl number of 10, 20 and 30 have shown in Figure 4. The boundary layer thickness and temperature decrease with an increase in Prandtl number as observed in Figure 4.

The variation in fluid concentration profile distribution θ(η) with respect to Lewis number (Le) of 10, 20 and 30 have been observed and presented in Figure 5.

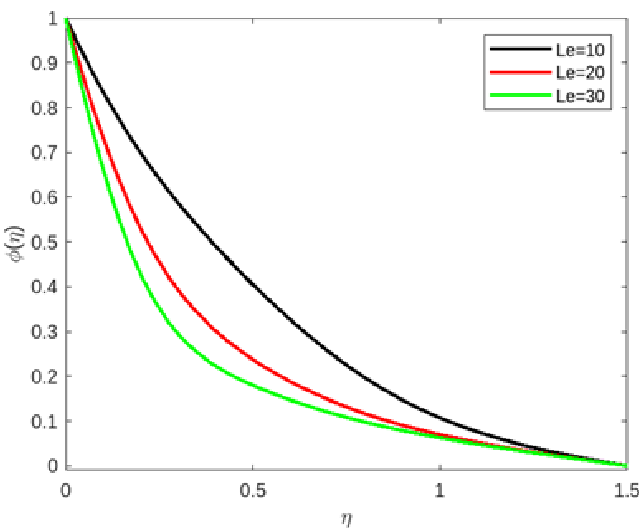


Figure 5. Variations in concentration profile ϕ(η)with Lewis number (Le).

Since the rate of mass transfer of nano fluid rises with rise of Lewis number value, the concentration decreases with an increase in thickness of boundary layer.

4.0 Conclusions

- The findings of current study may be applicable to various model investigations in various branches of science and technology where the surface layers are extended. The present study on nano fluid mass and heat flow transportation across stretching surface has been summarized as follows:
- The nano fluid momentum transfer, energy transfer and concentration depend mainly on four non dimensional parameters namely Prandtl number (*Pr*), Lewis number (*Le*), Brownian motion (*Nb*) and thermophoresis parameter (*Nt*).
- The fluid temperature rises for larger values of thermophoresis parameter (*Nt*) and Brownian motion (*Nb*) however found decreased with higher values of Prandtl number (*Pr*).
- The nanoparticle concentration profile declines with an intensification of Lewis number (Le) and Brownian motion (*Nb*).
- There was a rise and fall of reduced Sherwood number (Sh_r) and reduced Nusselt number (Nu_r) with varying thermophoresis parameter (*Nt*) at constant Prandtl number (*Pr*), Lewis number (*Le*), Brownian motion (*Nb*).
- The research reveals some insight on importance of nanofluids and boundary layer flow in aerospace industry for their structural parts, engine parts and landing gears.

5.0 Acknowledgment

The authors are thankful to B.M.S. College of Engineering, M.S. Ramaiah Institute of Technology, Sapthagiri College of Engineering and VTU for the necessary support extended to this work.

6.0 References

1. Takhar HS, Chamkha AJ, Nath G. Unsteady three-dimensional MHD boundary-layer flow due to the impulsive motion of a stretching surface. Acta Mech. 2001; 146:59–71.
2. Vleggaar J. Laminar boundary layer behaviour on continuous accelerating surface. Chem Eng Sci. 1977; 32:1517–1525.
3. Crane LG. Flow past a stretching plate. J appl math phys. 1970; 21:645–647.
4. Andersson HI, Bech KH, Dandapat BS. Magnetohydrodynamic flow of a power-law fluid over a stretching sheet. Int. J. Non-Linear Mech. 1992; 27(6):929-936.
5. Magyari E, Keller B. Exact solutions for self-similar boundary- layer flow induced by permeable stretching walls. Eur J Mech B Fluids. 2000; 19:109–122.
6. Sparrow EM, Abraham JP. Universal solutions for the stream wise variation of the temperature of a moving sheet in the presence of a moving fluid. Int J Heat Mass Transf. 2005; 48:3047–3056.
7. Kakac S, Pramuanjaroenkij A. Review of convective heat transfer enhancement with nano fluids. Int J Heat Mass Transf. 2009; 52:3187–3196.
8. Choi SUS, Eastman JA. Enhancing thermal conductivity of fluids with nanoparticles. IMECE. 1995; 66:99–105.
9. Choi SUS, Zhang ZG, Yu W, Lockwood FE, Grulke EA. Anomalously thermal conductivity enhancement in nanotube suspensions. Appl Phys Lett. 2001; 79:2252–2254.
10. Khanafer K, Vafai K, Lighthouse M. Buoyancy-driven heat transfer enhancement in a two-dimensional enclo-

sure utilizing nano fluids. Int J Heat Mass Transf. 2003; 46:3639–3653.

11. Kang HU, Kim SH, Oh JM. Estimation of thermal conductivity of nano fluid using experimental effective particle volume. Exp Heat Transf. 2006; 19:181–191.
12. Maiga SEB, Palm SJ, Nguyen CT, Roy G, Galanis N. Heat transfer enhancement by using nano fluids in forced convection flow. Int J Heat Fluid Flow. 2005; 26:530–546.
13. Tzou DY. Thermal instability of nano fluids in natural convection. Int J Heat Mass Transf. 2008; 51:2967–2979.
14. Hakan FO, Abunanda E. Numerical study of natural convection in partially heated rectangular enclosures filled with nano fluids. Int J Heat Fluid Flow. 2008; 29:1326–1336.
15. Buongiorno J. Convective transport in nano fluids. J Heat Transfer. 2006; 128:240–250.
16. Kuznetsov AV, Nield DA. Natural convective boundary-layer flow of a nano fluid past a vertical plate. Int J Thermal Sci. 2010; 49:243-247.
17. Kuznetsov AV, Nield DA. The Cheng-Minkowycz problem for natural convective boundary layer flow in a porous medium saturated by a Nano fluid. Int J Heat Mass Transf. 2013; 65:682-685.
18. Cheng P, Minkowycz WJ. Free convection about a vertical flat plate embedded in a porous medium with application to heat transfer from a dike. J Geophys Res.1977; 82:2040–2044.
19. Khan WA, Pop I. Boundary-layer flow of a nano fluid past a stretching sheet. Int J Heat Mass Transf. 2010; 53:2477–2483.
20. Ghasemi SE, Mohsenian S, Gouran S, Zolfagharian A. A novel spectral relaxation approach for Nano fluid flow past a stretching surface in presence of magnetic field and nonlinear radiation. Results Phys. 2022; 32:105141.
21. Samuel FM, Motsa SS. A highly accurate trivariate spectral collocation method of solution for two-dimensional nonlinear initial boundary value problems. Appl Math Comput. 2019; 360:221–235.
22 Raj N, Mondal S. Spectral methods to solve nonlinear problems. Partial Differential Equations in Applied Mathematics. 2021: 4:100043.
23. Seyed MM, Mohammadreza NR, Mohammad Y, Saeed D, Pop I, Mikhail AS. Dual solutions for Casson hybrid nanofluid flow due to a stretching/shrinking sheet: A new combination of theoretical and experimental models. Chin J Phys. 2021; 71:574-588.
24. Srinivasalu T, Shankar Goud B. Effect of inclined magnetic field on flow, heat and mass transfer of Williamson nanofluid over a stretching sheet. Case Stud Therm Eng. 2021; 23:100819.
25. Vishwanath BA, Maheshkumar N, Wakif A. Haar wavelet scrutinization of heat and mass transfer features during the convective boundary layer flow of a nanofluid moving over a nonlinearly stretching sheet. Partial Differential Equation in Applied Mathematics. 2021; 4:100192.
26. Kavya S, Nagendramma V. Opposing and assisting flow of a hybrid Newtonian /non-Newtonian nanofluid past a stretching cylinder. JMMF. 2023; 71(8):1045-1057.
27. Musharraf SM, Srinivas G. Performance evaluation of hypersonic flow past blunt bodies with Aerospikes using numerical techniques. JMMF. 2023; 71(2):186-195.