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An Analytical Treatment on Coriolis Force and Non-Uniform Temperature Gradient for Isotropic and Anisotropic Free Convection in Porous Channel

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Abstract

The effects of Coriolis force and non-uniform temperature gradient on the onset of convection in a horizontal rectangular channel filled up with an incompressible fluid fully saturated with anisotropic porous media is studied by means of linear stability analysis. We derive the critical Rayleigh number expression analytically, the assumptions made in this study are, the walls of the conducting channels are considered to be impermeable and are good thermal conductors to maintain un-uniform temperature gradient vertically. The solution of the problems is found using the method of Fourier series with the coefficients as functions of horizontal co-ordinate. It is found that the effect of rotation in anisotropic porous permeability and thermal diffusivity have a significant effect on the Rayleigh number and the steady flow patterns and the effect of rotation more often destabilises the system and the Critical Rayleigh number is determined.

Keywords: *Coriolis Force, Double Diffusion, Fourier Analysis*

1.0 Introduction

Comprehension of Fluid Mechanics principals has its utility in wide range of Applications ranging from enormous to tiny scale in diverse fields. Our work in this regard is an addition to the already existing body of knowledge accumulated through rigorous research and findings. The Thermally induced convection of fluids in porous media attracted lots of attention over past few decades because of its multiple utilities in the study of Geophysics along with various energy related systems. The study of linear convection in porous medium is of prominence in scientific studies undertaken due to its relevance in creating models simulating Geo thermal fields and insulation problems. Also, research on thermally-driven convection in porous media which are nonhomogeneous or anisotropic are important because of their practical real-world utilities which includes various naturally occurring phenomena. Examples include convection of ground water motion through sediments anisotropic rock beds of geothermal systems and the transfer of heat through porous or fibrous insulating materials of walls of buildings containing multi-layered walls windows and other air gaps in unventilated systems.

Thermal convection in a porous media is mainly based on the Darcy law which was stated by Darcy in 1856. Muskat was able to find a correlation between theory

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and experiment. After the findings of Muskat, there has been significant development in the study of flow through porous media[1]. Guckenheimer and Knobloch have investigated the two-dimensional convection in a horizontal layer of Bousinnesq fluid rotating about a vertical axis. They have found that for a certain few parameters, the dispersion relation describing the stability if the conductive solutions have two zero eigen values. The transition between the steady and oscillatory convection takes place as a function of Rayleigh and Taylor numbers that are near their respective critical values[2]. Patil and Kulkarni studied the effects of chemical reaction on free convective flow of a polar fluid through porous medium in the presence of internal heat generation[3]. Nield and Bejan have given an excellent summary of free convection flow in porous media[4]. Hassanien *et al*. studied the effects of thermal stratification on non-Darcy mixed convection from a vertical flat plate embedded in a porous medium[5]. Recently, Mohammadein and El-Shaer have studied combined free and forced convective flow past a semi-infinite vertical plate embedded in a porous medium incorporating the variable permeability[6]. The effect of Coriolis force on the convection in isotropic porous media has been investigated by Rudraiah and Srimani, Rudraiah and Rohini using linear theory[7]. the porous medium of small permeability and large rotation tend to stabilise the system. The results are shown graphically as well as numerically[8].

2.0 Objective and Scope

The Main objective of this thesis is to study the linear convection in porous media, in a horizontal rectangular channel of height 'h' and width 'a', filled up with an incompressible fluid fully saturated with anisotropic porous media. The Effect of Coriolis force is taken along z-direction and Non-Uniform Temperature gradient on the onset of convection and the Gravitational force is acting along z-direction. the assumptions made in this study are, the walls of the conducting channels are considered to be impermeable and are good thermal conductors to maintain un-uniform temperature gradient vertically. By employing the linear stability theory, in which the dependent variables are expanded in Fourier series with its coefficients being function of horizontal co-ordinate rather than vertical co-ordinate or time, we determine the critical Rayleigh number at the onset of convection and review the steady flow patterns for moderately super critical Rayleigh numbers.

3.0 Applications

A Rotating Biological Contactor (or RBC) is a cylindrical filter usually installed at a location; it is used to remove matter along with other chemicals like ammonia from water. RBCs are also used to remove Copper, silver, cadmium, zinc, nickel, tin and other heavy metals from wastewater[9]. Despite RBC's being uncommon when compared with organic trickling filters or oxidation ditches, the effluent and waste water output produced by them is of considerable quality. RBC consists of a cylindrical drum with discs mounted on the rotating shaft and biological slime is coated on the cylinder. Approximately 90% of the surface is immersed inside the waste water. The tanks are made to rotate, slime is rotated through the air and waste water so that the slime

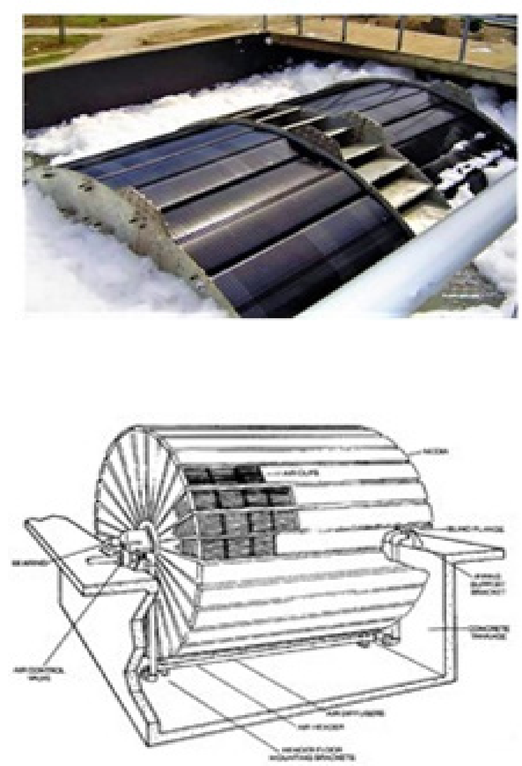

Figure 1. Rotating Biological Contactor.
Source: a. https://water.mecc.edu b. https://web.deu.edu.tr

takes up oxygen in the air and breaks down B.O.D. in the wastewater.

Likewise, the applications can be seen in the case of Hydropower plant, Reverse Osmosis and the effect of Coriolis force can be observed in the rotatory motion of the Centrifuge tubes used to separate the particles of Blood based on density. In this project, a miniature version of the flow and the phenomena happening is considered and worked on.

4.0 Mathematical Formulation

The three-dimensional thermal convection in incompressible fluid saturated porous channel, in a horizontal rectangular channel filled with anisotropic porous media is rotated in the vertical z-direction with angular velocity is considered. The walls of the channel are assumed to be impermeable and heat-conducting and heated nonuniformly to establish a nonuniform temperature gradient in the vertical z-direction. The channel is filled with incompressible fluid filled up by anisotropic porous media. The channel is rectangular with height 'h' and width 'a' and Cartesian coordinate system with the z axis in the vertical direction and x axis in the horizontal direction perpendicular to the channel axis is chosen. The horizontal channel walls are at z=0 and z=h. The vertical walls are at at x=-a/2 and x=a/2

The following are the governing equations for the given problem.

Conservation of Mass : (for incompressible fluids ρ = constant)

$$\nabla.\vec{q} = 0 \tag{1}$$

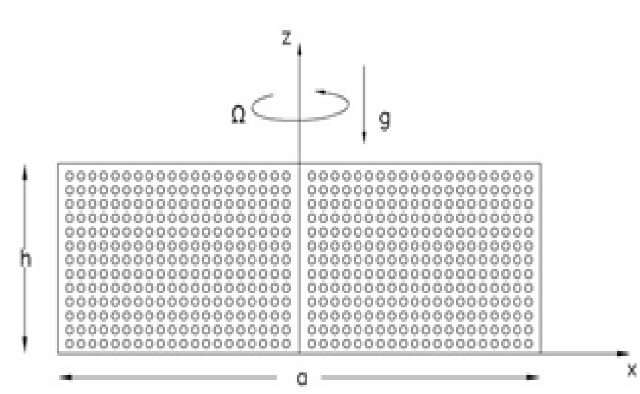

Figure 2. Physical configuration.

Conservation of Momentum:

$$\rho\left[\frac{\partial\vec{q}}{\partial t} + (\vec{q}.\nabla)\vec{q} + 2\Omega\hat{k}\ \times\vec{q}\right] = -\nabla p + \rho\hat{g} - \mu\vec{\tilde{q}} + \mu\nabla^2\vec{q} \tag{2}$$

where,

$$\vec{\tilde{q}} = \left(\frac{u}{k_x}, \frac{v}{k_y}, \frac{w}{k_z}\right),$$

Conservation of Energy:

$$c\,\frac{\partial T}{\partial t} + (\vec{q}.\nabla)T = \kappa_x\frac{\partial^2 T}{\partial x^2} + \kappa_y\frac{\partial^2 T}{\partial y^2} + \kappa_z\frac{\partial^2 T}{\partial z^2} \tag{3}$$

Equation of State:

$$\rho = \rho_0[1 - \beta(T - T_0)] \tag{4}$$

where, the Darcy velocity $q^{\rightarrow}$ is the volumetric flow rate of fluid per unit area,p is the pressure, ρ is the density, μ is viscosity, κ_x,κ_y,κ_z are thermal diffusivities. κ_x,κ_y,κ_z are permeabilities in x, y, z directions. T_0 is the reference temperature, T is the temperature at the point of fluid. Ω is the angular velocity of the system. $g^{\rightarrow}$ is the acceleration due to gravity.

The basic state of the system is given by

$$\vec{q} = (u, v, w) = (0,0,0), \vec{g} = (0,0,-g), \vec{\omega} = (0,0,\omega)$$

A fluctuation is also included which is a function of x, y and z and t.

In the perturbed state various quantities are given by

$$u = u'; v = v'; w = w'; \rho = \overline{\rho}(z) + \rho';$$

$$p = p(z) + p'; T = T(z) + T'$$

Therefore, $\overline{T}$ becomes,

$$\overline{T} = T_\circ + \Delta T\left(1 - \frac{z}{h}\right) \tag{5}$$

Since the static temperature distribution is independent of x, y and z the motionless conduction state exists. The scope of present study is limited to this case. Therefore, the temperature can be conveniently expressed in the form.

$$T = \overline{T} + \theta, \qquad T = \left[T_\circ + \Delta T\left(1 - \frac{z}{h}\right)\right] + \theta, \tag{6}$$

where, is the deviation from the static temperature. The perturbed equations in component wise form after removing primes are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

$$\rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - 2\Omega v\right] = -\frac{\partial p}{\partial x} - \frac{\mu}{k_x}u + \mu\nabla^2 u.$$

$$\rho\left[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + 2\Omega u\right] = -\frac{\partial p}{\partial y} - \frac{\mu}{k_y}v + \mu\nabla^2 v.$$

$$\rho\left[\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right] = -\frac{\partial p}{\partial z} - \frac{\mu}{k_z}w - \rho g + \mu\nabla^2 w$$

$$c\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} = \kappa_x\frac{\partial^2 T}{\partial x^2} + \kappa_y\frac{\partial^2 T}{\partial y^2} + \kappa_z\frac{\partial^2 T}{\partial z^2}$$

$$\rho = \rho_0[1 - \beta(T - T_0)], \tag{7}$$

where, μ is the viscocity of the porous medium. Here we make use of the well known Boussinesq approximation, that is, the variation density is neglected in all the terms of the momentum equation except only in the Buoyancy term. That is the product of variation of density and velocity is very small and is neglected, but in the buoyancy term the product of density and gravity is not small. Invoking the Boussinesq approximation, the equation (7) becomes,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - 2\Omega v = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} - \frac{\nu}{k_x}u + \nu\nabla^2 u,$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + 2\Omega u = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} - \frac{\nu}{k_y}v + \nu\nabla^2 v,$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

$$= -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - \frac{\nu}{k_z}w - \frac{\rho}{\rho_0}g + \nu\nabla^2 w,$$

$$c\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} = \kappa_x\frac{\partial^2 T}{\partial x^2} + \kappa_y\frac{\partial^2 T}{\partial y^2} + \kappa_z\frac{\partial^2 T}{\partial z^2}, \tag{8}$$

where, $\nu = \dfrac{\mu}{\rho_0}$ is the kinematic viscosity. When the flow is considered to be axisymmetric about y-axis, that is the partial derivatives of all the dependent variables with respect to y are zero. Therefore, the above equations reduces to

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} - 2\Omega v = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} - \frac{\nu}{k_x}u + \nu\nabla_1^2 u\,,$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + w\frac{\partial v}{\partial z} + 2\Omega u = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} - \frac{\nu}{k_y}v + \nu\nabla_1^2 v\,,$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - \frac{\nu}{k_z}w - \frac{\rho}{\rho_0}g + \nu\nabla_1^2 w\,,$$

$$c\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + w\frac{\partial T}{\partial z} = \kappa_x\frac{\partial^2 T}{\partial x^2} + \kappa_z\frac{\partial^2 T}{\partial z^2}, \tag{9}$$

where, $\nabla_1^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial z^2}.$

Since the flow is due to strong heating from below, the thermal diffusion is assumed to be larger than the viscous diffusion. Therefore, making an assumption that the Prandtl-Darcy number is large, we neglect the inertia and viscous terms from the momentum equation and let

$$v.\nabla T = u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial z}$$

is the nonlinear term in the conservation of energy equation. Since the flow axisymmetric, we introduce the stream function ψ=ψ(x,z) and u=∂ψ/∂z ; w=-∂ψ/∂x which will satisfy the continuity equation. Substituting the stream function ψ in above equation, we get

$$\frac{1}{\rho_\circ}\frac{\partial p}{\partial x} + \frac{\nu}{k_x}\frac{\partial \psi}{\partial z} - 2\Omega v = 0,$$

$$\frac{1}{\rho_0}\frac{\partial p}{\partial y} + \frac{\nu}{k_y}v + 2\Omega\frac{\partial \psi}{\partial z} = 0,$$

$$\frac{1}{\rho_0}\frac{\partial p}{\partial z} + \frac{\rho}{\rho_0}g - \frac{\nu}{k_z}\frac{\partial \psi}{\partial x} = 0,$$

$$c\frac{\partial T}{\partial t} + \frac{\partial \psi}{\partial z}\frac{\partial T}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial T}{\partial z} = \kappa_x\frac{\partial^2 T}{\partial x^2} + \kappa_z\frac{\partial^2 T}{\partial z^2}, \tag{10}$$

These equations are in dimensional form. Therefore, we non-dimensionalise these equations with the characteristic quantities or reference quantities by using the following non-dimensional variables

$$u=\frac{\kappa_z a u^*}{h^2} \quad v=\frac{\kappa_z a v^*}{h^2} \quad w=\frac{\kappa_z w^*}{h} \quad t=\frac{c h^2 t^*}{\kappa_z}$$

$$x=ax^* \quad y=ay^* \quad \psi=\frac{\kappa_z a\psi^*}{h} \quad \theta=\Delta T\theta^*$$

$$T_0=\Delta T T_0^* \quad p=\frac{v k_z \rho_0 p^*}{k_x},$$

where, * denotes the non-dimensional variables. By substituting these above non dimensional variables into equation (10), eliminating the pressure, density and * , then we get the governing equations describing the onset of convection in a three-dimensional rectangular channel rotating with angular velocity Ω as

$$\left(\xi\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)\psi+\xi R_a\frac{\partial\theta}{\partial x}-T_aD_a\frac{\partial v}{\partial z}=0, \tag{11}$$

$$\zeta\frac{\partial v}{\partial z}+T_aD_a\frac{\partial^2\psi}{\partial z^2}=0, \tag{12}$$

$$\left(\eta\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)\theta-\frac{\partial\psi}{\partial x}=\frac{\partial\theta}{\partial t}+v.\nabla\theta, \tag{13}$$

where, the Darcy-Rayleigh number R_a (also called Rayleigh number) is defined by

$$R_a=\frac{\beta g\Delta T k_z h}{\kappa_z \nu},$$

And ξ, ζ, η are the anisotropy aspect ratios of permeabilities and diffusivities of temperature

$$\xi=\frac{k_x}{k_z}\left(\frac{h}{a}\right)^2, \eta=\frac{\kappa_x}{\kappa_z}\left(\frac{h}{a}\right)^2, \ \zeta=\frac{k_x}{k_z},$$

The boundary conditions such that for a heat conducting and impermeable boundaries are

$$\psi=\theta=\frac{\partial v}{\partial z}= 0 \ on$$

$$\begin{cases} x=-\frac{1}{2}, & x=\frac{1}{2} \quad 0<z<1 \\ z=0, & z=1 \quad -\frac{1}{2}<x<\frac{1}{2} \end{cases} \tag{14}$$

5.0 Linear Stability and Flow Patterns

For the linear stability and steady flow patterns the nonlinear term $\underset{\sim}{v}.\nabla\theta=0$ therefore, the linearised form of the governing equations (11), (12) and (13) are

$$\left(\xi\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)\psi+\xi R_a\frac{\partial\theta}{\partial x}-T_aD_a\frac{\partial v}{\partial z}=0,$$

$$\xi\frac{\partial v}{\partial z}+T_aD_a\frac{\partial^2\psi}{\partial z^2}=0,$$

$$\left(\eta\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)\theta-\frac{\partial\psi}{\partial x}=\frac{\partial\theta}{\partial t}, \tag{15}$$

The inception of the convection is described by the linear versions of equations (3.20), (3.21) and (3.22). The solutions for ψ and θ is expanded in the Fourier series as

$$\psi=e^{\sigma t}\left[\frac{1}{2}C_0+\sum_{n=1}^{\infty} C_n(x)\cos n\pi z+D_n(x)\sin n\pi z\right],$$

$$\theta=e^{\sigma t}\left[\frac{1}{2}F_0+\sum_{n=1}^{\infty} F_n(x)\cos n\pi z+G_n(x)\sin n\pi z\right],$$

$$v=e^{\sigma t}\left[\frac{1}{2}S_0+\sum_{n=1}^{\infty} S_n(x)\cos n\pi z+H_n(x)\sin n\pi z\right] \tag{16}$$

$C_n, D_n, F_n, G_n, S_n \ and \ H_n$ are functions of x and σ is the growth rate. The boundary conditions (14) will be satisfied if $C_n=F_n=H_n=0$ for all values of x. Comparing the sin n πz terms for ψ and θ and cos n πz for v, the above equations can be written as

$$\psi=e^{\sigma t}D_n(x)\sin n\pi z,$$

$$\psi=e^{\sigma t}D_n(x)\sin n\pi z, \tag{17}$$

$$v=e^{\sigma t}S_n(x)\sin n\pi z.$$

By substituting the above Fourier series solutions equation (17) into the above linearized equations (15), then we get the following differential equations for the coefficients D_n, G_n and S_n,

$$\left(\xi\frac{d^2}{dx^2}-n^2\pi^2\right)D_n+\xi R_a\frac{dG_n}{dx}+T_aD_an\pi S_n=0$$

$$\zeta S_n+T_aD_an\pi D_n=0,$$

$$\left(\eta\frac{d^2}{dx^2}-n^2\pi^2\right)G_n-\frac{dD_n}{dx}=\sigma G_n, \tag{18}$$

The boundary conditions for D_n,G_n and S_n are given by

$$D_n\left(\frac{1}{2}\right) = D_n\left(-\frac{1}{2}\right) = 0$$

$$G_n\left(\frac{1}{2}\right) = G_n\left(-\frac{1}{2}\right) = 0,$$

$$S_n\left(\frac{1}{2}\right) = S_n\left(-\frac{1}{2}\right) = 0. \tag{19}$$

The operators on the left-hand side of the equations (11) to (12) with the given boundary condition (14) are shown to be self-adjoint, from this we can conclude that is real. Hence, to obtain critical Rayleigh number R_{ac}, which is function of ξ,η,ζ. For marginal stability, we put σ =0, the above equations (18) then read as

$$\left(\xi\,\frac{d^2}{dx^2} - n^2\,\pi^2\right)D_n + \xi R_a\frac{dG_n}{dx} + T_aD_an\pi S_n = 0,$$

$$\zeta S_n + T_an\pi D_n = 0,$$

$$\left(\eta\,\frac{d^2}{dx^2} - n^2\pi^2\right)G_n - \frac{dD_n}{dx} = 0, \tag{20}$$

And the equations (17) become

$$\psi = D_n\sin n\,\pi z,$$

$$\theta = G_n\sin n\,\pi z,$$

$$\nu = S_n\cos n\,\pi z. \tag{21}$$

The system of equations (18) are subjected to the boundary conditions and will represent the self adjoint Eigen value problem with Eigen value R_a and with R_{ac} being the smallest. The problem under consideration can be assumed in two cases

i) Isotropic case where ζ=η=ξ This case is easy to solve and the critical Rayleigh number can be found easily.

ii) Anisotropic case where ζ≠η≠ξ. In this case we find the Rayleigh number numerically as the analytical method proves to be more tedious and the Rayleigh number can be found in the hyperbola in the ζ,η plane. The critical Rayleigh number is also determined.

Case 1: Isotropic Case

Along with isotropic medium the condition ζ=η=ξ is satisfied when $\dfrac{k_x}{k_z} = \dfrac{\kappa_x}{\kappa_z} = \dfrac{k_x}{k_y}$ that is the ratio of the horizontal and vertical components of the permeability and the thermal diffusivity are equal. Hence, (18) reduces to

$$\left(\xi\,\frac{d^2}{dx^2} - n^2\,\pi^2\right)D_n + \xi R_a\frac{dG_n}{dx} + T_aD_an\pi S_n = 0. \tag{22}$$

$$\xi S_n + T_an\pi D_n = 0 \tag{23}$$

$$\left(\xi\,\frac{d^2}{dx^2} - n^2\pi^2\right)G_n - \frac{dD_n}{dx} = 0 \tag{24}$$

Using the equations (23) and (24), we eliminate G_n (x) and S_n (x) from equation (22) we get equation forD_n

$$\left[\xi\,\frac{d^2}{dx^2} - n^2\pi^2\left(1 + \frac{T_a^2D_a^2}{\xi}\right)\right]\left[\xi\,\frac{d^2}{dx^2} - \,n^2\pi^2\right]D_n + \xi R_a\frac{d^2D_n}{dx^2} = 0 \tag{25}$$

This equation (25) together with the boundary condition an eigenvalue problem with the eigenvalue R_a. Now the roots of the equation (25) are obtained by replacing λ=d/dx in the above equation, for the eigen value 'λ' as

$$\left[\xi\lambda^2 - n^2\pi^2\left(1 + \frac{T_a^2D_a^2}{\xi}\right)\right][\xi\lambda^2 - n^2\pi^2]D_n + \xi R_a\lambda^2D_n = 0$$

By solving the equation we get a quadratic equation, finding the roots of the quadratic equation we get the value of λ as

$$\lambda = \pm\frac{i}{2\sqrt{\xi}}\sqrt{R_a - (n\pi)^2\left(2 + \frac{T_a^2D_a^2}{\xi}\right) + 2(n\pi)^2\sqrt{1 + \frac{T_a^2D_a^2}{\xi}}} \pm \sqrt{R_a - (n\pi)^2\left(2 + \frac{T_a^2D_a^2}{\xi}\right) - 2(n\pi)^2\sqrt{1 + \frac{T_a^2D_a^2}{\xi}}} \tag{26}$$

The general solution for $D_n(x)$ is

$$D_n(x, R_a) = C_1 \cos px + C_2 \sin px + C_3 \cos qx + C_4 \sin qx, \tag{27}$$

$$p = \frac{1}{2\sqrt{\xi}}\sqrt{R_a - (n\pi)^2\left(2 + \frac{T_a^2 D_a^2}{\xi}\right) + 2(n\pi)^2\sqrt{1 + \frac{T_a^2 D_a^2}{\xi}}} + \sqrt{R_a - (n\pi)^2\left(2 + \frac{T_a^2 D_a^2}{\xi}\right) - 2(n\pi)^2\sqrt{1 + \frac{T_a^2 D_a^2}{\xi}}}$$

$$q = \frac{1}{2\sqrt{\xi}}\sqrt{R_a - (n\pi)^2\left(2 + \frac{T_a^2 D_a^2}{\xi}\right) + 2(n\pi)^2\sqrt{1 + \frac{T_a^2 D_a^2}{\xi}}} - \sqrt{R_a - (n\pi)^2\left(2 + \frac{T_a^2 D_a^2}{\xi}\right) - 2(n\pi)^2\sqrt{1 + \frac{T_a^2 D_a^2}{\xi}}} \tag{28}$$

We must see that the square root of the quantities involved must be real roots in particular and, C_1, C_2, C_3 and C_4 are constants and p, q are the roots of the equation (27). Now, substituting for $D_n(x)$ from equation (27) into equation (24) and (23), we find $G_n(x)$ and $S_n(x)$ by solving the equation

$$G_n(x, R_a) = s[rC_1 \sin px - rC_2 \cos px + C_3 \sin qx - C_4 \cos qx]$$

where,

$$r = \frac{q\,\xi p^2 + n^2\pi^2\left(1 + \frac{T_a^2 D_a^2}{\xi}\right)}{p\,\xi q^2 + n^2\pi^2\left(1 + \frac{T_a^2 D_a^2}{\xi}\right)},\ s = \frac{\xi q^2 + n^2\pi^2\left(1 + \frac{T_a^2 D_a^2}{\xi}\right)}{\xi R_a q},$$

$$S_n = \frac{-n\pi T_a D_a}{\xi}[C_1 \cos px + C_2 \sin px + C_3 \cos qx + C_4 \sin qx], \tag{30}$$

we consider the solution of the problem in two cases. Case(i) : when $C_2 = C_4 = 0$, the equations (27),(29) and (30) for D_n,G_n and S_n

$$D_n(x) = C_1 \cos px + C_3 \cos qx,$$

$$G_n(x) = s[rC_1 \sin px + C_3 \sin qx]$$

$$S_n(x) = -\frac{T_a D_a n\pi}{\xi}[C_1 \cos px + C_3 \cos qx]$$

We consider the solution of the problem in two cases Case(i) : when $C_2 = C_4 = 0$, the equations (27),(29) and (30) for D_n,G_n and S_n

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$$G_n(x) = s[rC_1 \sin px + C_3 \sin qx]$$

$$S_n(x) = -\frac{T_a D_a n\pi}{\xi}[C_1 \cos px + C_3 \cos qx]$$

The equations can be expressed in the matrix form and the non-trivial solution of C_1 and C_3 can be obtained

$$\frac{1}{2}\left[\sin\left(\frac{p+q}{2}\right) - \sin\left(\frac{p-q}{2}\right)\right] - \left(\frac{r}{2}\right)\left[\sin\left(\frac{p+q}{2}\right) + \sin\left(\frac{p-q}{2}\right)\right] = 0$$

$$(1-r)\sin\left(\frac{p+q}{2}\right) - (1+r)\sin\left(\frac{p-q}{2}\right) = 0 \tag{31}$$

when C_1=C_3=0, the equations (27), (29) and (30) for D_n, G_n and S_n reduces to

$$D_n(x) = C_2 \sin px + C_4 \sin qx$$

$$G_n(x) = -s[rC_2 \cos px + C_4 \cos qx],$$

$$S_n(x) = -\frac{T_a\, D_a n\pi}{\xi}[C_2 \sin px + C_4 \sin qx]$$

After substituting the boundary conditions,

$$C_2 \sin \frac{p}{2} + C_4 \sin \frac{q}{2} = 0,$$

$$r\, C_2 \cos \frac{p}{2} + C_4 \cos \frac{q}{2} = 0.$$

Similarly, the above equations can be written in the matrix form and the non-trivial solution of C_2 and C_4 can be obtained

$$\sin\frac{p}{2}\cos\frac{q}{2}-r\cos\frac{p}{2}\sin\frac{q}{2}=0,$$

$$(1-r)\sin\left(\frac{p+q}{2}\right)+(1+r)\sin\left(\frac{p-q}{2}\right)=0 \tag{32}$$

For isotropic porous media, ζ=η=ξ, r=1 and the equations (31) and (32) leads to the equation

$$\sin\left(\frac{p-q}{2}\right)=0$$

This implies that p-q=2mπ, m=1,2,3, ...
Substituting the value of p and q we get,

$$p-q=\frac{\sqrt{R_a-(n\pi)^2\left(2+\frac{T_a^2D_a^2}{\xi}\right)-2(n\pi)^2\sqrt{1+\frac{T_a^2D_a^2}{\xi}}}}{\sqrt{\xi}}=2m\pi,$$

Squaring and simplifying, we get

$$R_a=4\pi^2\left[\xi m^2+\frac{n^2}{2}\left(1+\frac{T_a^2\,D_a^2}{2\xi}+\sqrt{1+\frac{T_a^2\,D_a^2}{\xi}}\right)\right],$$

where, n=1, 2, 3… and m=1, 2, 3…
We get the smallest Eigen value for n = 1while m = 1 gives the Critical Rayleigh number Ra_c given by

$$R_{a_c}=4\pi^2\left[\xi+\frac{1}{2}\left(1+\frac{Ta^2\,D_a^2}{2\xi}+\sqrt{1+\frac{T_a^2\,D_a^2}{\xi}}\right)\right] \tag{33}$$

By further simplification, substituting the value of anisotropic ratio we get

$$R_{ac}=4\pi^2\left[1+\frac{1}{2}\left(1+\frac{Ta^2\,D_a^2}{2}+\sqrt{1+T_a^2\,D_a^2}\right)\right] \tag{34}$$

The onset of convection will characterize the flow for moderately super Critical Rayleigh numbers. The equations will coincide when ζ=η=ξ that is when r = 1 and we arrive at two linearly independent solutions for the problem. This can also be found from the stationary linearized versions of the equations (11), (12) and (13).

If the solutions at Ra=Ra_c substituting the value of Ra in P and q we get linearly independent solutions which are

$$p=\pi\left(\sqrt{1+\frac{1}{\xi}\sqrt{1+\frac{T_a^2\,D_a^2}{\xi}}+1}\right)=K+\pi$$

$$q=\pi\left(\sqrt{1+\frac{1}{\xi}\sqrt{1+\frac{T_a^2\,D_a^2}{\xi}}-1}\right)=K-\pi$$

where, $$K=\pi\sqrt{1+\frac{1}{\xi}\sqrt{1+\frac{T_a^2\,D_a^2}{\xi}}} \tag{35}$$

When C1 = C3 = 0 substituting the value of p and q to find D_n G_n and S_n to find $\psi^{(1)},\theta^{(1)},\nu^{(1)}$ we get,

$$D_n=C_2\sin\,px-\frac{C_2\sin\,\frac{p}{2}}{\sin\,\frac{q}{2}}\sin\,qx$$

$$G_n=-s\left[C_2\cos\,px-\frac{C_2\sin\,\frac{p}{2}}{\sin\,\frac{q}{2}}\cos\,qx\right]$$

$$S_n=-\frac{T_aD_a\pi C_2}{\xi}\left[\sin\,px-\frac{\sin\,\frac{p}{2}}{\sin\,\frac{q}{2}}\sin\,qx\right]$$

Substituting for p and q in terms of K and in the above equation we get

$$D_n=2C_2\,\sin K\,x\,\cos\pi\,x$$

$$G_n=-2sC_2\,\cos K\,x\,\cos\pi\,x$$

$$S_n=-\frac{2T_aD_a\pi C_2}{\xi}\,\sin K\,x\,\cos\pi\,x,$$

We write this solution as $\psi^{(1)},\theta^{(1)},\nu^{(1)}$ given as below

$$\psi^{(1)}=Q\sin\left(\pi\sqrt{1+\frac{1}{\xi}\sqrt{1+\frac{T_a^2\,D_a^2}{\xi}}}\,x\right)\cos\pi\,x\,\sin\pi\,z,$$

$$\theta^{(1)} = -Qs\cos\left(\pi\sqrt{1+\frac{1}{\xi}\sqrt{1+\frac{T_a^2D_a^2}{\xi}}}x\right)\cos\pi x\sin\pi z,$$

$$\nu^{(1)} = -\frac{T_aD_a\pi}{\xi}Q\sin\left(\pi\sqrt{1+\frac{1}{\xi}\sqrt{1+\frac{T_a^2D_a^2}{\xi}}}x\right)\cos\pi x\cos\pi z$$

(36)

Similarly, when C2 = C4 = 0 substituting the value of p find D_n G_n and S_n to find $\psi^{(2)}$,$\theta^{(2)}$,$\nu^{(2)}$ we get,

$$\psi^{(2)} = \frac{S}{s}\cos\left(\pi\sqrt{1+\frac{1}{\xi}\sqrt{1+\frac{T_a^2D_a^2}{\xi}}}x\right)\cos\pi x\sin\pi z,$$

$$\theta^{(2)} = S\,Sin\left(\pi\sqrt{1+\frac{1}{\xi}\sqrt{1+\frac{T_a^2D_a^2}{\xi}}}x\right)\cos\pi x\sin\pi z,$$

$$\nu^{(2)} = -\frac{T_aD_aS\pi}{\xi s}\cos\left(\pi\sqrt{1+\frac{1}{\xi}\sqrt{1+\frac{T_a^2D_a^2}{\xi}}}x\right)\cos\pi x\cos\pi z$$

(37)

Equation (37) represents the solution for isotropic case which is linearly independent at R_a=R_{ac}. Here C_2=C_4=0 and Q=2C_1 and S=2sC_1 are the amplitude constants. The solution of (36) yields symmetric flow pattern of 2n cells and n depends on ξ .

Case 2: Anisotropic Case

The equations for anisotropic case are

$$\left(\xi\frac{d^2}{dx^2}-n^2\pi^2\right)D_n+\xi R_a\frac{dG_n}{dx}+T_aD_an\pi S_n=0,$$

$$\zeta S_n+T_aD_an\pi D_n=0,$$

$$\left(\eta\frac{d^2}{dx^2}-n^2\pi^2\right)G_n-\frac{dD_n}{dx}=0$$

(38)

To find R_{ac} we need to solve the above equations with the boundary conditions and we will be able to eliminate D_n (x) G_n (x) S_n (x) and we get roots,

$$\lambda = \pm\frac{i}{2\sqrt{\xi\eta}}\sqrt{\begin{matrix}\xi R_a-(n\pi)^2\left[\xi+\eta\left(1+\frac{T_a^2D_a^2}{\zeta}\right)\right]\\ +2(n\pi)^2\sqrt{\xi\eta\left(1+\frac{T_a^2D_a^2}{\zeta}\right)}\end{matrix}}$$
$$\pm\sqrt{\xi R_a-(n\pi)^2\left[\xi+\eta\left(1+\frac{T_a^2D_a^2}{\zeta}\right)\right]-2(n\pi)^2\sqrt{\xi\eta\left(1+\frac{T_a^2D_a^2}{\zeta}\right)}}$$

(39)

The general solution for D_n (x) is

$$D_n(x,R_a)=C_1\cos\,px+C_2\sin\,px+C_3\cos\,qx+C_4\sin\,qx \quad (40)$$

where C_1,C_2,C_3,C_4 are arbitrary constants and the Eigen roots. The roots of the equartion p and q are given by

$$p=\frac{1}{2\sqrt{\xi\eta}}\sqrt{\xi R_a-(n\pi)^2\left[\xi+\eta\left(1+\frac{T_a^2D_a^2}{\zeta}\right)\right]+2(n\pi)^2\sqrt{\xi\eta\left(1+\frac{T_a^2D_a^2}{\zeta}\right)}}+$$
$$\sqrt{\xi R_a-(n\pi)^2\left[\xi+\eta\left(1+\frac{T_a^2D_a^2}{\zeta}\right)\right]-2(n\pi)^2\sqrt{\xi\eta\left(1+\frac{T_a^2D_a^2}{\zeta}\right)}}$$

$$q=\frac{1}{2\sqrt{\xi\eta}}\sqrt{\xi R_a-(n\pi)^2\left[\xi+\eta\left(1+\frac{T_a^2D_a^2}{\zeta}\right)\right]+2(n\pi)^2\sqrt{\xi\eta\left(1+\frac{T_a^2D_a^2}{\zeta}\right)}}$$
$$-\sqrt{\xi R_a-(n\pi)^2\left[\xi+\eta\left(1+\frac{T_a^2D_a^2}{\zeta}\right)\right]-2(n\pi)^2\sqrt{\xi\eta\left(1+\frac{T_a^2D_a^2}{\zeta}\right)}}$$

(41)

Calculation are carried out similar to that of isotropic case to find R_{ac}

$$R_{a_c}=\pi^2\left[\left(1+\sqrt{\frac{\eta}{\xi}\left(1+\frac{Ta^2D_a^2}{\zeta}\right)}\right)^2+4\eta\right],$$

(42)

The equation (42) which gives the criteria for the onset of convection in the case of Anisotropic porous media. Furthermore, the onset of convection can be characterised by supercritical Rayleigh number at R_a=R_{ac} there exists two sets of linearly independent solutions when C_1=C_3=0 and C_2=C_4=0, are obtained as follows:

The eigen roots p and q when, can be written as

$$p = \pi\left(\sqrt{1+\sqrt{\frac{1}{\xi\eta}\left(1+\frac{T_a^2\,D_a^2}{\zeta}\right)}}\right)+\pi = K+\pi$$

$$q = \pi\left(\sqrt{1+\sqrt{\frac{1}{\xi\eta}\left(1+\frac{T_a^2\,D_a^2}{\zeta}\right)}}\right)-\pi = K-\pi$$

where, $K = \pi\sqrt{1+\sqrt{\frac{1}{\xi\eta}\left(1+\frac{T_a^2\,D_a^2}{\xi}\right)}},$

When C1 = C3 = 0 substituting the value of p and q to find D_n G_n and S_n to find $\psi^{(1)},\theta^{(1)},\nu^{(1)}$ we get,

$$\psi^{(1)} = 2C_2\sin\left(\pi\sqrt{1+\sqrt{\frac{1}{\xi\eta}\left(1+\frac{T_a^2\,D_a^2}{\xi}\right)}}\,x\right)\cos\pi x\,\sin\pi z,$$

$$\theta^{(1)} = -2sC_2\left((+1)\cos\left(\pi\sqrt{1+\sqrt{\frac{1}{\xi\eta}\left(1+\frac{T_a^2\,D_a^2}{\zeta}\right)}}\,x\right)\cos\pi x\right)+$$

$$2sC_2\left(\begin{matrix}(1-r)\\ \sin\left(\pi\sqrt{1+\sqrt{\frac{1}{\xi\eta}\left(1+\frac{T_a^2\,D_a^2}{\zeta}\right)}}\,x\right)\cos\pi x\end{matrix}\right)\sin\pi z$$

$$\nu^{(1)} = -\frac{2T_aD_a\pi}{\zeta}C_2\sin\left(\pi\sqrt{\left\{1+\sqrt{\frac{1}{\eta\xi}\left(1+\frac{T_a^2\,D_a^2}{\xi}\right\}}x\right)}\right)\cos\pi x\,\cos\pi z$$

Similarly, when C2 = C4 = 0 substituting the value of p and q to find D_n G_n and S_n to find $\psi^{(2)},\theta^{(2)},\nu^{(2)}$ we get,

$$\psi^{(2)} = 2\frac{C_1}{s}\cos\left(\pi\sqrt{1+\sqrt{\frac{1}{\xi\eta}\left(1+\frac{T_a^2D_a^2}{\zeta}\right)}}\;x\right)\cos\pi x\;\sin\pi z$$

$$\theta^{(2)} = sC_1\left((1+r)\sin\left\{\pi\sqrt{1+\sqrt{\frac{1}{\xi\eta}\left(1+\frac{T_a^2D_a^2}{\zeta}\right)}}\,x\right\}\right)\cos\pi x$$

$$+sC_1\big((1-r)\cos K\,x\sin\pi\,x\big)\sin\pi\,z$$

$$\nu^{(2)} = -\frac{T_aD_a\pi S}{\zeta}\cos\left(\pi\sqrt{1+\sqrt{\frac{1}{\xi\eta}\left(1+\frac{T_a^2D_a^2}{\zeta}\right)}}\;x\right)$$

$\cos\pi\,x\cos\pi\,z$

6.0 Graphs and Isotherms

Isotropic case

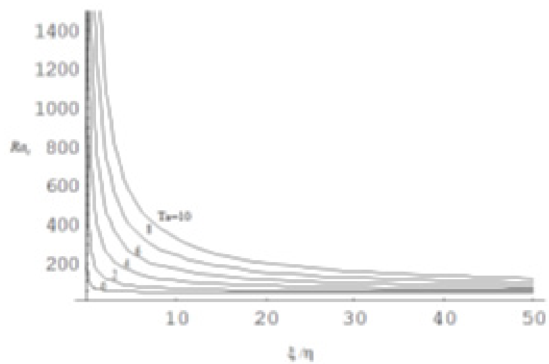

Figure 3. Graph (isotropic case).

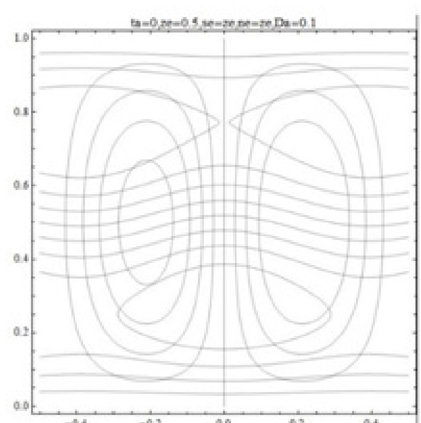

Figure 4. Isotherm 1 (isotropic case).

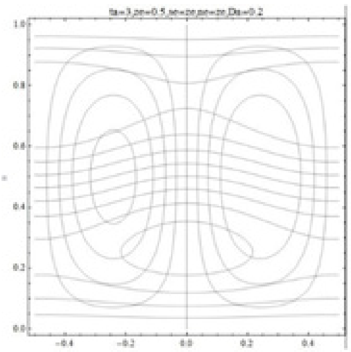

Figure 5. Isotherm 2 (Isotropic case).

Anisotropic case

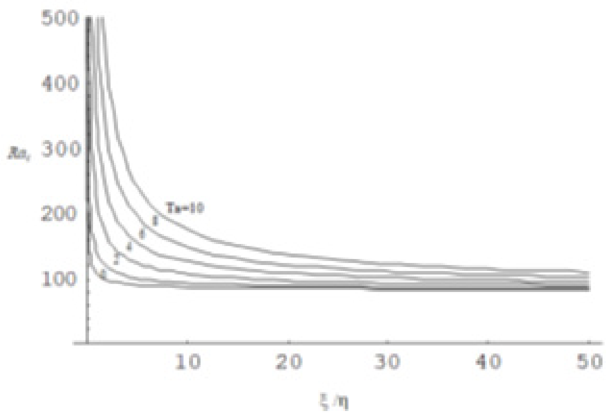

Figure 6. Graph (Anisotropic case).

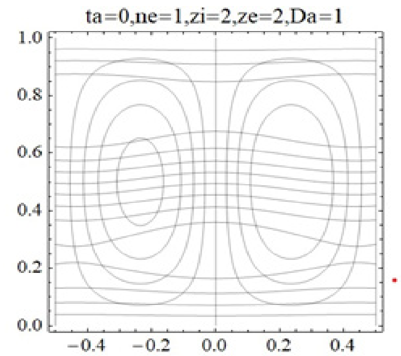

Figure 7. Isotherm 1 (anisotropic case).

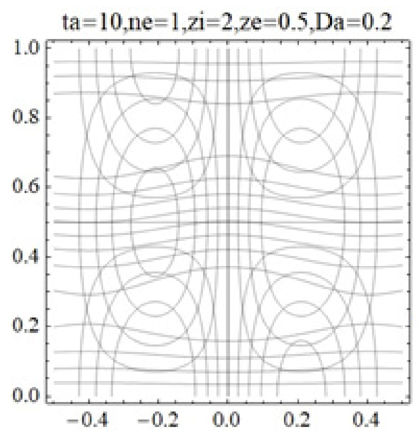

Figure 8. Isotherm 2 (anisotropic case).

7.0 Observations

- The number of cells in both cases were found to be increasing as the Taylor number increased.
- The increase in the flow behaviour is indicated with the thermal lines in the case of rotation and isotropy.
- The number of cells decrease with increase in the aspect ratio and the thermal diffusivity in anisotropic case.
- The critical Rayleigh number is inversely proportional to the ratio of ξ/η
- It also increases with the increase in the Taylor number, solutal Rayleigh number and with the effect of rotation.

8.0 Acknowledgement

The authors would firstly like to dedicate the project to their parents and also thank R. Sumithra and P. A. Dinesh for mentoring the project and encouraging and motivating to push ourselves even further.

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